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Spectral transfer morphisms for unipotent affine Hecke algebras

Eric Opdam[1]

Dedicated to Joseph Bernstein on the occasion of his 70th birthday, with admiration

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Abstract We classify the spectral transfer morphisms (cf. Opdam in Adv Math 286:912–957, 2016) between affine Hecke algebras associated to the unipotent types of the various inner forms of an unramified absolutely simple algebraic group G defined over a non-archimedean local field k. This turns out to characterize Lusztig's classification (Lusztig in Int Math Res Not 11:517–589, 1995; in Represent Theory 6:243–289, 2002) of unipotent characters of G in terms of the Plancherel measure, up to diagram automorphisms. As an application of these results, the spectral correspondences associated with such morphisms (Opdam 2016), and some results of Ciubotaru, Kato and Kato [CKK] (also see Ciubotaru and Opdam in A uniform classification of the discrete series representations of affine Hecke algebras. arXiv:1510.07274) we prove a conjecture of Hiraga, Ichino and Ikeda [HII] on formal degrees and adjoint gamma factors in the special case of unipotent discrete series characters of inner forms of unramified simple groups of adjoint type defined over k.

Keywords Hecke algebra · Formal degree · Spectral transfer morphism · L-packet

Mathematics Subject Classification Primary 20C08; Secondary 22D25, 43A30

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1 Introduction

Recall from [54] that a normalized affine Hecke algebra $\mathcal{H}$ is essentially determined by a complex torus T and a meromorphic function μ on T. A spectral transfer morphism (see [54]) $\phi : \mathcal{H}_1 \rightsquigarrow \mathcal{H}_2$ between normalized affine Hecke algebras expresses the fact that μ_1 is equal to a residue of μ_2 along a certain coset of a subtorus of T_2. This turns out to be a convenient tool to compare formal degrees of discrete series representations of different affine Hecke algebras.

The notion is based on the special properties of the μ-function of an affine Hecke algebra [52,53] which are intimitely related to its basic role in the derivation of the Plancherel formula for affine Hecke algebras via residues [21,52,55,56]. This approach to the computation of formal degrees has its origin in the theory of spherical functions for p-adic reductive groups [47], and was further inspired by early observations of Lusztig [35,38] and Reeder [59,60] on the behaviour of formal degrees within unipotent L-packets.

In the present paper we classify the spectral transfer morphisms (STMs in the sequel) between the unipotent affine Hecke algebras of the various inner forms of a given absolutely simple algebraic group G of adjoint type, defined and unramified over a non-archimedean local field $\mathbf{k}$. In particular we will show, for any unipotent type $\tau = (\mathbb{P}, \sigma)$ of an inner form of G, *existence* and *uniqueness* (up to diagram automorphisms) of such STM of the Hecke algebra of τ to the Iwahori–Hecke algebra $\mathcal{H}^{IM}(G)$ of G. The STMs of this kind turn out to correspond exactly to the arithmetic-geometric correspondences of Lusztig [40,43].

As an application of this classification, using the basic properties of STMs discussed in [54], we prove the conjecture [26, Conjecture 1.4] of Hiraga, Ichino and Ikeda expressing the formal degree of a discrete series representation in terms of the adjoint gamma factor of its (conjectural) local Langlands parameters and an explicit rational constant factor, for all unipotent discrete series representations of inner forms of G (where we accept Lusztig's parameters for the unipotent discrete series representations as conjectural Langlands parameters). It should be mentioned that it was already known from Reeder's work [59,60] (see also [26]) that this conjecture holds for the unipotent discrete series characters of split exceptional groups of adjoint type, and for some small rank classical groups. It should be mentioned that the stability of Lusztig's packets of unipotent representations was shown by Moeglin and Waldspurger for odd orthogonal groups [49] and by Moeglin for unitary groups [48].

Throughout this paper we use the normalization of Haar measures as in [17]. Let $q = v^2$ denote the cardinality of the residue field of $\mathbf{k}$. The formal degree of a unipotent discrete series representation then factorizes uniquely as a product of a q-rational number (which we define as a fraction of products of q-numbers of the form $[n]_q := \frac{(v^n - v^{-n})}{v - v^{-1}}$ with $n \geq 2$) and a positive rational number. Our proof of conjecture [26, Conjecture 1.4] involves the verification of the q-rational factors, which rests on the existence of the Plancherel measure preserving correspondences for STMs as discussed in [54], and the verification of the rational constants. The latter uses the knowledge of these rational constants from [60] for the case of equal parameter exceptional Hecke algebras, and continuity principles due to [11] and [56] (also [13]) which imply roughly that we can compute these rational constant factors in the formal degrees of discrete series of non-simply laced affine Hecke algebras at any point in the parameter space of the affine Hecke algebra once we know these rational constants in one *regular* point (in the sense of [56]) of the parameter space. In particular, for classical affine Hecke algebras of type $\mathrm{C}_n^{(1)}$; it was shown in [11] that at a generic point in the parameter space, the rational constants for all generic families of discrete series characters are equal. The constants at special parameters follow then by a continuity principle in the formal degree due to [56].

An alternative approach to the conjecture [26, Conjecture 1.4], restricted to the case of formal degrees of unipotent discrete series representations, was formulated in [12]. A conjectural formula for the formal degrees of unipotent discrete series characters is proposed in [12], which involves Lusztig's non-abelian Fourier transform matrix for families of unipotent representations [36,44,45] and a notion of the "elliptic fake degree" of a unipotent discrete series character in the unramified minimal principal series of G. In this approach the formula for the rational constant factors of the formal degrees appears in a very natural way from the basic properties of the non-abelian Fourier transform.

The notion of spectral transfer morphism is based on a certain heuristic idea on the behavior of L-packets under ordinary parabolic induction (see 3.1.3 for a more detailed discussion of this heuristic idea). The fact that this principle turns out to hold for all unipotent representations is striking. Also striking is the fact that the isomorphism class of the Iwahori–Hecke algebra $\mathcal{H}^{IM}(G)$ of G is the least element in the poset of isomorphism classes of normalized affine Hecke algebras in the full subcategory of $\mathfrak{C}_{\mathrm{es}}(G)$ whose objects are the Hecke algebras of unipotent types $(\mathbb{P}, \sigma)$ of the inner forms of G, in the sense of [54, Paragraph 7.1.5]. Moreover, if $\mathcal{H}$ is such a unipotent affine Hecke algebra of an inner form of G, then the STM $\phi : \mathcal{H} \rightsquigarrow \mathcal{H}^{IM}(G)$ (which exists by the above) is essentially unique, and such STMs exactly match Lusztig's arithmetic/geometric correspondences. The proof of these statements reduces, as explaned in this paper, to the supercuspidal case [19] in combination with the above principle that one can parabolically induce unipotent supercuspidal STMs from Levi subalgebras to yield new STMs.

It is quite clear that the definition of the notion of STM could be generalized to Bernstein components [2,23–25] in greater generality than only for the unipotent Bernstein components. It would be interesting to investigate the above mentioned induction principle in general. In view of our results, this could provide a clue how L-packets are partitioned by the Bernstein center beyond Lusztig's unipotent L-packets for simple groups of adjoint type.

In the first section of this paper we will review the theory of unipotent representations of G with an emphasis on its harmonic analytic aspects. The results here are all due to [40,43,50,51] and [17]. This section serves an important purpose of reviewing the relevant facts on unipotent representations for this paper in the appropriate context of harmonic analysis, and fixing notations. We kept the setup in this section more general than necessary for the remainder of the paper, since this does not complicate matters too much and this may be useful for later applications. In the second section we will describe the structure of the STMs between the normalized unipotent Hecke algebras of the inner forms of G, and discuss the applications of this result.[1]

2 Unipotent representations of quasisimple p-adic groups

The category of unipotent representations of inner forms of an unramified absolutely quasisimple p-adic group G is Morita equivalent to the category of representations of a finite direct sum of finitely many normalized affine Hecke algebras (called "unipotent Hecke algebras") in such a way that the Morita equivalence respects the tempered spectra and the natural Plancherel measures on both sides.

Therefore it is an interesting problem to classify all the STMs as defined in [54, Definition 5.1] between these unipotent normalized affine Hecke algebras. It will turn out that this task to classify these STMs essentially reduces to the task of finding all STMs from the rank 0 unipotent affine Hecke algebras to the Iwahori–Matsumoto Hecke algebra $\mathcal{H}^{IM}(G')$ of the quasisplit G' such that G is an inner form of G'. In turn this reduces to solving equation [54, equation (55)] where d^0 denotes the formal degree of a unipotent supercuspidal representation. The latter part of this task, the classification of the rank 0 unipotent STMs, will be discussed in a second paper (joint with Yongqi Feng [19]). It should be remarked that the results of the present paper, in which the existence of certain spectral transfer morphisms is established, plays a role in the proof of the classification result in [19].

2.1 Unramified reductive p-adic groups

Let k be non-archimedean local field. Fix a separable algebraic closure $\overline{k}$ of k, and let $K \subset \overline{k}$ be the maximal unramified extension of k in $\overline{k}$. Let $\mathbb{K} = \mathcal{O}/\mathcal{P}$ be the residue field of K, and let p denote its characteristic. Let $\Gamma = \mathrm{Gal}(\overline{k}/k)$ denote the absolute

[1] It is a pleasure to thank Joseph Bernstein, Dan Ciubotaru, Maarten Solleveld, David Kazhdan and Mark Reeder for useful discussions and comments.

Galois group of k, and let $\mathcal{I} = \mathrm{Gal}(\overline{k}/K) \subset \Gamma$ be the inertia subgroup. Let Frob be the geometric Frobenius element of $\mathrm{Gal}(K/k) = \Gamma/\mathcal{I} \simeq \hat{\mathbb{Z}}$, i.e., the topological generator which induces the *inverse* of the automorphism $x \to x^{\mathbf{q}}$ of $\mathbb{K}$. Here $\mathbf{q} = p^n$ denotes the cardinality of the residue field $\mathbf{k} := \mathbb{K}^{\mathrm{Frob}}$ of k. We denote by $\mathbf{v}$ the positive square root of $\mathbf{q}$.

Let $\mathbf{G}$ be a connected reductive algebraic group defined over k, and split over K. We denote by $G^\vee$ be the neutral component of a Langlands dual group ${}^LG$ for G (see [3]). The construction of ${}^LG$ presupposes the choice of a maximal torus $\mathbf{S}$ and a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ whose Levi-subgroup is $\mathbf{S}$, and the choice of an épinglage for $(\mathbf{G}, \mathbf{B}, \mathbf{S})$, in order to define a splitting of $\mathrm{Aut}(\mathbf{G})$. Let $X^*(Z(G^\vee))$ be the character group of the center $Z(G^\vee)$ of $G^\vee$. The natural Γ-action on this space factors through the quotient $\mathrm{Gal}(K/k)$ since we are assuming that G is K-split. Observe that the action of Frob on $X^*(Z(G^\vee))$ is independent of the choice of a splitting of $\mathrm{Aut}(\mathbf{G})$.

We will always denote the group $\mathbf{G}(K)$ of K-rational points of $\mathbf{G}$ by the corresponding non-boldface letter, i.e., $G = \mathbf{G}(K)$. Kottwitz [31, Section 7] has defined a Γ-equivariant functorial exact sequence

$$1 \to G_1 \to G \xrightarrow{w_G} X^*(Z(G^\vee)) \to 1. \tag{1}$$

In our situation there is a continuous equivariant action of the group $\Gamma/\mathcal{I}$ on this sequence. We denote by F the action of Frob on G_1 and G, and by θ_F the automorphism of $X^*(Z(G^\vee))$ defined by F. This sequence has the property that the associated long exact sequence in continuous nonabelian cohomology yields an exact sequence

$$1 \to {G_1}^{\mathrm{F}} \to G(k) \to X^*(Z(G^\vee))^{\langle\theta_F\rangle} \to 1 \tag{2}$$

and an isomorphism

$$H^1(F, G) \xrightarrow{\sim} X^*(Z(G^\vee))_{\langle\theta_F\rangle}. \tag{3}$$

Now assume that G is semisimple. In this situation the above sequences simplify as follows. Let $\mathbf{S}$ be a maximal K-split torus of $\mathbf{G}$, and let $X := X_*(\mathbf{S})$ be its cocharacter lattice. Let $Q := X_{\mathrm{sc}} = X_*(\mathbf{S}_{\mathrm{sc}})$ be the cocharacter lattice of the inverse image of $\mathbf{S}$ in the simply connected cover $\mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ of $\mathbf{G}$ (hence $Q \subset X$ is the coroot lattice of $(\mathbf{G}, \mathbf{S})$; we warn the reader that we call the roots of $G^\vee$ "roots" and the roots of $(\mathbf{G}, \mathbf{S})$ "coroots". We apologize for this admittedly awkward convention). Let Ω be the finite abelian group $\Omega = X/Q$. Then we may canonically identify $X^*(Z(G^\vee))$ with Ω. Hence (2) becomes

$$1 \to {G_1}^{\mathrm{F}} \to G(k) \to \Omega^{\theta_F} \to 1 \tag{4}$$

(see [29,30]) and (3) becomes

$$H^1(F, G) \xrightarrow{\sim} \Omega/(1 - \theta_F)\Omega. \tag{5}$$

We remark that $G_{\mathrm{der}} \subset G_1 \subset G$, and that it can be shown that $G_{\mathrm{der}} = G_1$ if and only if p does not divide the order $|\Omega|$ of Ω. *We will from now on always assume that* $\mathbf{G}$ *is absolutely quasisimple and* K*-split, unless otherwise stated.*

2.1.1 Inner k-rational structures of G The k-rational structures of $\mathbf{G}$ which are inner forms of G are parameterized by $H^1(k, \mathbf{G}_{\mathrm{ad}})$. By Steinberg's Vanishing Theorem it follows that all inner k-forms of G are K-split and that $H^1(k, \mathbf{G}_{\mathrm{ad}}) = H^1(\mathrm{Gal}(K/k), G_{\mathrm{ad}})$ (see [62, Section 5.8]). *We will from now on reserve the notation G for a k-quasisplit rational structure in this inner class.* We let F be the automorphism of G_{ad} (or G) corresponding to the action of Frob, and $\theta = \theta_F$. We then denote the nonabelian cohomology $H^1(\mathrm{Gal}(K/k), G_{\mathrm{ad}})$ by $H^1(F, G_{\mathrm{ad}})$.

For G semisimple and not necessarily of adjoint type, Vogan conjectured a refined Langlands parameterization of the irreducible tempered unipotent representations of *pure inner forms* of G [68].

Pure inner form of G correspond by definition to cocycles $z \in Z^1(F, G)$ [17,68]. Such a cocycle is determined by the image $u := z(\mathrm{Frob}) \in G$. The corresponding inner k-form of G is defined by the functorial image $z^{\mathrm{ad}} \in Z^1(F, G_{\mathrm{ad}})$ of z. This "pure" inner form is defined by the twisted Frobenius action F_u on G given by $F_u = \mathrm{Ad}(u) \circ F$, and is denoted by G^u. The cocycle z determines a class in $[z] \in H^1(F, G)$. We say that two pure inner forms z_1 and z_2 of G are equivalent iff $[z_1] = [z_2]$. The k-rational isomorphism class of the inner form G^u is determined by the image $[z^{\mathrm{ad}}]$ of $[z]$ via the natural map $H^1(F, G) \to H^1(F, G_{\mathrm{ad}})$. The reader be warned however, that view of (5) this map is neither surjective in general (this is obvious, $G = SL_2$ provides an example) nor injective (however, if G is k-split and semisimple, then the map is injective). In other words, not all k-rational equivalence classes of inner forms of G can be represented by a pure inner form, and if G is not k-split and semisimple, then an inner form of G may be represented by several inequivalent pure inner forms.

It is in principle possible to compute with our methods the formal degrees of the elements of L-packets according to this refined form of the Langlands parameterization, or even to check examples of the conjecture [26, Conjecture 1.4] beyond the case of pure inner forms. For later reference we will formulate matters in this more general setup where possible, even though we will in present paper limit ourselves in the applications to the case where G is of adjoint type.

2.1.2 The affine Weyl group There exists a maximal K-split torus $\mathbf{S}$ defined over k and maximally k-split [5, 5.1.10]. We fix such a maximal torus S of G, and denote by S_{sc} its inverse image for the covering $G_{\mathrm{sc}} \to G$. Recall that G is k-quasisplit, and that F defines an automorphism on the lattices X and $X_{\mathrm{sc}} = Q$ denoted by θ. The extended affine Weyl group W of (G, S) is defined by

$$W = N_G(S)/S_{\mathcal{O}}. \tag{6}$$

The group W acts faithfully on the apartment $\mathcal{A}$ as an extended affine Coxeter group.

We denote by $S_{\mathcal{O}} = \mathcal{O}^\times \otimes X$ the maximal bounded subgroup of S. Then $X = S/S_{\mathcal{O}}$, and we define the associated F-stable apartment $\mathcal{A} = \mathcal{A}(G, S)$ of the building of G by $\mathcal{A}(G, S) = \mathbb{R} \otimes X$. As explained in [17, Corollary 2.4.3], [6, Section 3] the isomorphism (5) can be made explicit by a canonical bijection

$$\Omega/(1-\theta)\Omega \xrightarrow{\sim} H^1(F, G) \tag{7}$$

sending $[\omega] \in \Omega/(1-\theta)\Omega$ to the cohomology class of the cocycle z_u which maps Frob to F_u, where $uS_{\mathcal{O}} = x \in X$ and x is a representative of $\omega \in X/Q$.

Let C be an F-stable alcove in $\mathcal{A}$ (such alcoves exist, see [67]). Let $1 \to \mathbf{N} \to \mathbf{G}_{\mathrm{sc}} \to \mathbf{G} \to 1$ be the simply connected cover of $\mathbf{G}$, and let $\mathbf{S}_{\mathrm{sc}}$ be the inverse image of $\mathbf{S}$.

Proposition 2.1 *The image of $G_{sc} \to G$ is equal to the derived group G_{der} of G, and we have $G/G_{der} \xrightarrow{\sim} H^1(K, \mathbf{N}) = K^\times \otimes \Omega$.*

Proof Indeed, it is clear that the image is contained in G_{der} because G_{sc} is its own derived group [65]. The other inclusion follows by applying the long exact sequence in nonabelian cohomology to the central isogeny $\mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ and again appealing to Steinberg's Vanishing Theorem. It follows that the quotient of G by the image of G_{sc} is the abelian group $H^1(K, \mathbf{N})$, whence the result. On the other hand, we have the obvious exact sequence

$$1 \to \mathrm{Hom}(\Omega^*, K^\times) \to S_{\mathrm{sc}} \to S \to K^\times \otimes \Omega \to 1 \tag{8}$$

which we can compare to the long exact sequence in cohomology (with respect to $\mathcal{I}$) associated to the canonical exact sequence of diagonalizable groups $1 \to \mathbf{N} \to \mathbf{S}_{\mathrm{sc}} \to \mathbf{S} \to 1$. □

We denote by W^a_C the F-stable normal subgroup of W generated by the reflections in the walls of C. This normal subgroup is independent of the choice of C and can be canonically identified with $N_{G_{\mathrm{der}}}(S)/S_{\mathcal{O}} \cap G_{\mathrm{der}} \xrightarrow{\sim} W^a \subset W$, the affine Weyl group of $(G_{\mathrm{sc}}, S_{\mathrm{sc}})$.

Returning to Kottwitz's homomorphism we obtain the following result (compare with [5, 5.2.11]).

Corollary 2.2 *We have $G_1 = \langle S_{\mathcal{O}}, G_{der} \rangle$.*

Proof Let $\mathbb{B}$ be the Iwahori subgroup of G associated with C [5, 5.2.6]. By [57, Appendix, Proposition 3] we have $\mathbb{B} = \mathrm{Fix}(C) \cap G_1$. In particular we have $S_{\mathcal{O}} \subset G_1$, so that we have $G_{\mathrm{der}} \subset G_1' := \langle S_{\mathcal{O}}, G_{\mathrm{der}} \rangle \subset G_1$ Hence by (4), the equality $G_1' = G_1$ is equivalent to showing that $G/G_1' = G/G_1 = \Omega$. By the previous proposition we have $G/G_{\mathrm{der}} = K^\times \otimes \Omega$. Since $S_{\mathcal{O}}/S_{\mathcal{O}} \cap G_{\mathrm{der}} = \mathcal{O}^\times \otimes \Omega$ the result follows from $K^\times/\mathcal{O}^\times \simeq \mathbb{Z}$. □

Let Ω_C be the subgroup of W which stabilizes C. This subgroup may be identified with a subgroup of the group of special automorphisms (in the sense of [40, paragraph 1.11]) of the affine diagram associated with the choice of C. We have a semidirect product decomposition $W = W^a \rtimes \Omega_C$, and thus a canonical isomorphism $\Omega_C \xrightarrow{\sim} \Omega$ for any choice of C.

Corollary 2.3 *We have $N_{G_1}(S)/S_{\mathcal{O}} \xrightarrow{\sim} W^a$, $N_{G_1}(\mathbb{B}) = \mathbb{B}$ and $N_G(\mathbb{B})/\mathbb{B} \xrightarrow{\sim} \Omega$.*

Proof By Corollary 2.2 it follows that $N_{G_1}(S) = N_{G_{\mathrm{der}}}(S).S_{\mathcal{O}}$. This implies the first assertion, since W^a is the affine Weyl group of $(G_{\mathrm{sc}}, S_{\mathrm{sc}})$ and G_{der} is the homomorphic image of G_{sc}. Since an Iwahori-subgroup of G_{sc} is self-normalizing, we have similarly $N_{G_1}(\mathbb{B}) = N_{G_{\mathrm{der}}}(\mathbb{B}).S_{\mathcal{O}} = (\mathbb{B} \cap G_{\mathrm{der}}).S_{\mathcal{O}} = \mathbb{B}$, proving the second assertion. For the third assertion, observe that $\Omega_C = (N_G(\mathbb{B}) \cap N_G(S))/S_{\mathcal{O}}$. It is well known that $\mathbb{B} \cap N_G(S) = S_{\mathcal{O}}$, hence Ω_C maps injectively into $N_G(\mathbb{B})/\mathbb{B}$. By the second assertion this group maps injectively into $G/G_1 = \Omega$. Since $\Omega \simeq \Omega_C$ are finite the two injective homomorphisms are in fact isomorphisms.

Since G is unramified there exist hyperspecial points in the apartment $\mathcal{A}$ [67]. A choice of a hyperspecial point $a_0 \in \mathcal{A}$, induces a semidirect product decomposition $W = W_0 \ltimes X$, where W_0 denotes the isotropy subgroup of a_0 in W. The k-structure of G defined by F is quasisplit, which implies that there exists a hyperspecial point $a_0 \in \mathcal{A}(G, S)$ which is F-fixed. In this case we denote by θ the automorphism of W (and of $\mathcal{A}$) induced by F. We fix a_0, an F-fixed hyperspecial point, and an F-stable alcove C having a_0 in its closure. Observe that the subgroup Ω_C depends on the choice of C, not of the hyperspecial point a_0. Recall we have a canonical isomorphism $\Omega_C \xrightarrow{\sim} \Omega = X/Q$, which we will often use to identify these two groups. Observe that θ stabilizes the subgroups W^a, Ω_C, X and W_0 of W.

2.2 Unipotent representations

2.2.1 Parahoric subgroups Recall the explicit representation of pure inner forms G^u as discussed in (7). Fix a representative $u = \dot{\omega} \in N_G(S)$ with $\omega \in \Omega_C \subset W$. Then F_u acts on the apartment $\mathcal{A}(G, S)$ by means of the finite order automorphism $\omega\theta$. Since F_u stabilizes C the Iwahori subgroup $\mathbb{B}$ is F_u stable. Recall that the group Ω_C can be canonically identified with the group $N_G(\mathbb{B})/\mathbb{B}$. Since Ω is abelian it is clear that the subgroup $\Omega_C^{F_u} = \Omega_C^{\omega\theta}$ of F_u-invariant elements is independent of $\omega \in \Omega_C$.

Following [57, Appendix] we may define a "standard parahoric subgroup of G" as a subgroup of the form $\mathrm{Fix}(F_P) \cap G_1$ where $F_P \subset C$ denotes a facet of C. By [57, Appendix, Proposition 3] this definition coincides with the definition in [5]. In particular, a standard parahoric subgroup of G is a connected pro-algebraic group. A parahoric subgroup of G is a subgroup conjugate to a standard parahoric subgroup.

It is well known (by "Lang's theorem for connected proalgebraic groups", see [40, 1.3]) that any F_u-stable parahoric subgroup of G is G^{F_u}-conjugate to a "standard" F_u-stable parahoric subgroup, i.e., an F_u-stable parahoric subgroup containing $\mathbb{B}$. It follows that the G^{F_u}-conjugacy classes of F_u-stable parahoric subgroups are in one-to-one correspondence with the set of Ω^θ-orbits of $\omega\theta$-stable facets in the closure of C. Similarly, a parahoric subgroup $\mathbb{P}$ or a double coset of a parahoric subgroup is F_u-stable iff it contains points of G^{F_u}. Let $\mathbb{P}$ be an F_u-stable parahoric subgroup of G. We call $\mathbb{P}^{F_u}$ a parahoric subgroup of G^{F_u}.

We record two important properties of F_u-stable parahoric subgroups which follow easily from Corollary 2.3. First of all, parahoric subgroups are self-normalizing in G_1, i.e.,

$$(N_G\mathbb{P})^{F_u} \cap G_1 = \mathbb{P}^{F_u}. \tag{9}$$

Secondly, for an F_u-stable standard parahoric subgroup $\mathbb{P}$ corresponding to a $\omega\theta$-stable facet C_P of C, we have

$$(N_G\mathbb{P})^{F_u}/\mathbb{P}^{F_u} = \Omega^{\mathbb{P},\theta} \tag{10}$$

where $\Omega^{\mathbb{P}} \subset \Omega_C$ is the subgroup stabilizing C_P, and $\Omega^{\mathbb{P},\theta} \subset \Omega^{\mathbb{P}}$ its fixed point group for the action of θ (or $F_u = \omega\theta$, which amounts to the same since Ω_C is abelian). We define an exact sequence

$$1 \to \Omega_1^{\mathbb{P},\theta} \to \Omega^{\mathbb{P},\theta} \to \Omega_2^{\mathbb{P},\theta} \to 1 \tag{11}$$

where $\Omega_1^{\mathbb{P},\theta}$ is the subgroup of elements which fix the set of F_u-orbits of vertices of C not in C_P pointwise.

2.2.2 Normalization of Haar measures Let G, F, and F_u be as in the previous paragraph. Then G^{F_u} is a locally compact group. For any F_u-stable parahoric subgroup $\mathbb{P}$ of G we denote by $\overline{\mathbb{P}^{F_u}}$ the reductive quotient of $\mathbb{P}^{F_u}$. This is the group of $\mathbf{k}$-points of a connected reductive group over $\mathbf{k}$. In particular this is a finite group.

Following [17, Section 5.1] we normalize the Haar measure of G^{F_u} uniquely, such that for all F_u-stable parahoric subgroups $\mathbb{P}$ of G one has

$$\mathrm{Vol}(\mathbb{P}^{F_u}) = \mathbf{v}^{-a}|\overline{\mathbb{P}^{F_u}}| \tag{12}$$

where $a \in \mathbb{Z}$ is equal to the dimension of $\overline{\mathbb{P}}$ over $\mathbb{K}$. It is well known that the right-hand side is a product of powers of $\mathbf{v}$ and cyclotomic polynomials in $\mathbf{v}$.

2.2.3 The anisotropic case It is useful to discuss the case where G^{F_u} is anisotropic explicitly. It is well known that an anisotropic absolutely simple group G^{F_u} is isomorphic to $\mathrm{PGL}_1(\mathbb{D}) := \mathbb{D}^\times/k^\times$, where $\mathbb{D}$ is an unramified central division algebra over k of degree $m+1$, rank $(m+1)^2$ (see for instance [16]). We choose a uniformizer π of k. $\mathbb{D}$ contains an unramified extension l of degree $m+1$ over k, and we may choose a uniformizer Π of $\mathbb{D}$ which normalizes l, such that conjugation by Π restricted to l yields a generator for $\mathrm{Gal}(l:k)$, and such that $\Pi^{m+1} = \pi$. The group $\mathbb{P} := G_1$ is the only F_u-stable parahoric subgoup in this situation, and obviously $G = N\mathbb{P}$. By (4) we have $\Omega := G^{F_u}/G_1^{F_u} \approx \frac{\langle\Pi\rangle}{\langle\pi\rangle} \approx \mathrm{Gal}(l:k)$, a cyclic group of order $m+1$. G^{F_u} contains a maximal prounipotent subgroup G_+ (denoted by V_1 in [16]) and we have $G^{F_u} = C.G_+$, where C is generated by the anisotropic torus $T^{F_u} := l^\times/k^\times$ and Π. We see that the reductive quotient $\mathbb{P}/(\mathbb{P}\cap G_+)$ is an anisotropic torus $\overline{\mathbb{T}}$ of rank m over $\mathbf{k}$, and that $\overline{\mathbb{T}^{F_u}}$ can be identified with the group of roots of unity of order prime to p in l modulo the subgroup of those roots of unity in k. Hence $\mathrm{Vol}(G^{F_u}) = v^{-m}|\Omega||\overline{\mathbb{T}^{F_u}}| = (m+1)[m+1]_q$ (with $[m+1]_q$ the q-integer associated to $m+1 \in \mathbb{N}$ [see Definition 2.6)].

2.2.4 Unipotent representations and affine Hecke algebras Let G be a quasisimple linear algebraic group, defined and quasisplit over k and K-split as above. Recall that the automorphism induced by the Frobenius F on the building of G was denoted by

θ. Recall that the inner forms of G are canonically parameterized by the abelian group $\Omega/(1-\theta)\Omega$. Let $u = \dot{\omega} \in N\mathbb{B}$ be a representative of an element $\bar{\omega} \in \Omega/(1-\theta)\Omega$ and let F_u denote the corresponding pure inner twist of F. We denote by G^u the pure inner form of G defined by this twisted k-structure (in particular $G^1 = G$).

A representation (E, δ) of a parahoric subgroup $\mathbb{P}^{F_u}$ of G^{F_u} (where $\mathbb{P}$ is an F_u-stable parahoric subgroup of G) is called *cuspidal unipotent* if it is the lift to $\mathbb{P}^{F_u}$ of a cuspidal unipotent representation of the reductive quotient $\overline{\mathbb{P}^{F_u}}$. An F_u-stable parahoric subgroup is called cuspidal unipotent if it has cuspidal unipotent representations.

Lusztig [40] introduced the category $R(G^{F_u})_{\mathrm{uni}}$ of unipotent representations of G^{F_u}. A smooth representation (V, π) of G^{F_u} is called unipotent if V is generated by a sum of cuspidal unipotent isotypical components of restrictions of (V, π) to various parahoric subgroups of G^{F_u}. As a generalization of Borel's theorem on Iwahori-spherical representations, $R(G^{F_u})_{\mathrm{uni}}$ is an abelian subcategory of the category $R(G^{F_u})$ of smooth G^{F_u} representations. It is central to the approach in this paper that this category is equivalent to the module category of an explicit finite direct sum of normalized Hecke algebras in the sense of paragraph [54, 3.1.2], in a way which is compatible with harmonic analysis. Let us therefore describe this in detail.

A cuspidal unipotent pair $(\mathbb{P}, \delta)$ consists of an F_u-stable parahoric subgroup $\mathbb{P}$ of G and an irreducible cuspidal unipotent representation δ of $\mathbb{P}^{F_u}$. We say that $(\mathbb{P}, \delta)$ is standard if $\mathbb{P}$ is standard. Let $R(G^{F_u})_{(\mathbb{P},\delta)}$ denote the subcategory of $R(G^{F_u})$ consisting of the smooth representations (V, π) such that V is generated by the isotypical component $(V|_{\mathbb{P}^{F_u}})_\delta$. According to [40], given two cuspidal unipotent pairs $(\mathbb{P}_i, \delta_i)$ (with $i \in \{1, 2\}$) the subcategories $R(G^{F_u})_{(\mathbb{P}_i,\delta_i)}$ are either disjoint or equal, and this last alternative occurs if and only if the pairs $(\mathbb{P}_i, \delta_i)$ are G^{F_u}-conjugates (and not just associates). It follows that a smooth representation (V, π) is unipotent iff V is generated by $\oplus(V|_{\mathbb{P}^{F_u}})_\delta$, where the direct sum is taken over a complete set of representatives $(\mathbb{P}, \delta)$ of the finite set of Ω^θ-orbits of standard cuspidal unipotent pairs.

For each standard cuspidal unipotent pair $\mathfrak{s} = (\mathbb{P}, \delta)$ we consider the algebra $\mathcal{H}_{\mathbf{v}}^{u,\mathfrak{s}}$ of $\mathfrak{s}$-spherical $\mathrm{End}(E)$-valued functions on G^{F_u}, equipped with a trace $\tau(f) := \mathrm{Tr}_V(f(e))$ and $*$ defined by $f^*(x) := f(x^{-1})^*$. This algebra turns out to be the specialization at $\mathbf{v}$ of a finite direct sum of mutually isomorphic normalized (in the sense of [54, paragraph 3.1.2]) affine Hecke algebras (called unipotent affine Hecke algebras) defined over $\mathbf{L} = \mathbb{C}[v^{\pm 1}]$, and has been explicitly determined in all cases [40,51]. The following general result from the theory of types due to [7] (also see [22]) is fundamental to the approach in this paper:

Theorem 2.4 *The assignment $(V, \pi) \to V^{(\mathbb{P},\delta)} := Hom_{\mathbb{P}}(\delta, V|_{\mathbb{P}^{F_u}})$ establishes an equivalence of categories from $R(G^{F_u})_{(\mathbb{P},\delta)}$ to the category of $\mathcal{H}_{\mathbf{v}}^{u,\mathfrak{s}}$-modules which respects the notion of temperedness and which is a Plancherel measure preserving on the level of irreducible tempered representations.*

2.2.5 The group of weakly unramified characters Recall that we have a canonically identification of Ω with $N_G(\mathbb{B})/\mathbb{B}$.

By the application of Lang's Theorem for proalgebraic groups [40, paragraph 1.8] one sees that the F_u stable double $\mathbb{B}$-cosets Θ in G are precisely those which intersect with G^{F_u}, in which case $\Theta \cap G^{F_u}$ is a single double coset of $\mathbb{B}^{F_u}$.

Let $u = \dot{\omega}$ be a representative of an element $\omega \in \Omega/(1-\theta)\Omega$. We see that the double $\mathbb{B}^{F_u}$-cosets of G^{F_u} are parameterized by the $\omega\theta$ fixed group $W^{\omega\theta}$, and that $\Omega^{\omega\theta} = N_{G^{F_u}}\mathbb{B}^{F_u}/\mathbb{B}^{F_u}$. Because Ω is abelian we actually have $\Omega^{\omega\theta} = \Omega^{\theta}$. We have $W^{\omega\theta} = W^{\omega\theta,a} \rtimes \Omega^{\theta}$. By [40] this extended affine Weyl group is the underlying affine Weyl group of the Iwahori–Hecke algebra $\mathcal{H}^{u,IM} := \mathcal{H}^{u,(\mathbb{B},1)}$ of the group G^{F_u}. When $[\omega] = 1$ we denote this algebra simply by $\mathcal{H}^{IM}$, the *generic* Iwahori–Hecke algebra.

By (4), the Pontryagin dual $(\Omega^{\theta})^*$ of Ω^{θ} can be viewed canonically as the group of (weakly) unramified complex linear characters $X^*_{\mathrm{un}}(G^{F_u})$ of G^{F_u} (i.e., the complex linear characters of G^{F_u} vanishing on $G_1^{F_u}$). This defines a natural functorial action of $(\Omega^{\theta})^*$ on the category $R(G^{F_u})_{\mathrm{uni}}$ (by taking tensor products). These functors are Plancherel measure preserving, as we will see, and play an important role.

2.3 Unramified local Langlands parameters

The based root datum of the connected component $G^{\vee}$ of the Langlands dual group ${}^LG$ of G^u is defined by $\mathcal{R} = (X, R_0, Y, R_0^{\vee}, F_0)$. The dual Langlands group of G^u is independent of u and defined by

$$ {}^LG := G^{\vee} \rtimes \langle\theta\rangle \tag{13} $$

where θ denotes the outer automorphism of $G^{\vee}$ arising from F. Let $S^{\vee} \subset G^{\vee}$ be a maximal torus of $G^{\vee} = G^{\vee}(\mathbb{C})$. Let $Z(G^{\vee})$ be the center of the neutral component $G^{\vee}$ of ${}^LG$. Then $Z(G^{\vee}) \simeq P^{\vee}/Y = \Omega^* \subset S^{\vee}$. We will denote by ${}^LZ$ the central subgroup ${}^LZ := Z(G^{\vee})^{\theta} \subset {}^LG$, so that ${}^LZ$ is canonically equal to $(\Omega^*)^{\theta} \subset S^{\vee,\theta}$. It follows that we can canonically identify the group $\Omega/(1-\theta)\Omega$ with the Pontryagin dual group of ${}^LZ$, which is the version of Kottwitz's Theorem as explained in detail in [17].

Let us recall the space of unramified local Langlands parameters for G^{F_u} for later reference. Let $\mathcal{W}_k$ denote the Weil group of k [66], with inertia subgroup $\mathcal{I} \subset \mathcal{W}_k$, and let Frob denote a generator of $\mathcal{W}_k/\mathcal{I}$. An *unramified local Langlands parameter* is a homomorphism

$$ \lambda : \langle \mathrm{Frob} \rangle \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG \tag{14} $$

such that $\lambda(\mathrm{Frob} \times \mathrm{id}) = s \times \theta$ (with $s \in G^{\vee}$) semisimple and such that λ is algebraic on the $\mathrm{SL}_2(\mathbb{C})$-factor. Given an unramified Langlands parameter λ we denote by $[\lambda]$ its orbit for the action of $G^{\vee}$ by conjugation. We will write Λ for the set of orbits $[\lambda]$ of unramified Langlands parameters.

If λ is an unramified Langlands parameter, let $A_{\lambda} := \pi_0(C_{G^{\vee}}(\lambda))$ be the component group of the centralizer of λ in $G^{\vee}$. We call λ *elliptic* (or discrete) if $C_{G^{\vee}}(\lambda)$ is finite, and denote by Λ^e the space of $G^{\vee}$-orbits of unramified elliptic Langlands parameters.

Let λ be an unramified elliptic Langlands parameter. Observe that ${}^LZ = Z(G^{\vee})^{\theta} \subset Z(G^{\vee}) \subset A_{\lambda}$. The inner forms G^u of G are canonically parameterized via Kottwitz's Theorem by the character group of ${}^LZ_{\mathrm{sc}}$, the center of the L-group of $G_{\mathrm{ad}} = G/Z(G)$. Given an inner form G^u, we choose once and for all a character $\zeta_u \in \mathrm{Irr}(Z_{\mathrm{sc}})$ (with $Z_{\mathrm{sc}} := Z(G^{\vee})_{\mathrm{sc}})$) which restricts to the character $\omega_u \in \Omega_{\mathrm{ad}}/(1-\theta)\Omega_{\mathrm{ad}}$ of ${}^LZ_{\mathrm{sc}}$ that is represented by $u = \dot{\omega} \in N_{G_{\mathrm{ad}}}(\mathbb{B})$. Following [20, Section 7.2] (see also [1]) we consider the group $A_\lambda/Z(G^\vee) \subset (G^\vee)_{\mathrm{ad}}$, and let $\mathcal{A}_\lambda \subset (G^\vee)_{\mathrm{sc}}$ be its full preimage in the simply connected cover $(G^\vee)_{\mathrm{sc}}$ of $(G^\vee)_{\mathrm{ad}}$. Thus $Z_{\mathrm{sc}} \subset \mathcal{A}_\lambda$, and $\mathcal{A}_\lambda$ is a central extension of $A_\lambda/Z(G^\vee)$ by Z_{sc}. We denote by $\mathrm{Irr}^u(\mathcal{A}_\lambda)$ the set of irreducible characters ρ of $\mathcal{A}_\lambda$ on which Z_{sc} acts by a multiple of ζ_u.

The space of ($G^\vee$-orbits of) unramified discrete Langlands data for G^u is defined by

$$\tilde{\Lambda}^u := \{(\lambda, \rho) \mid [\lambda] \in \Lambda^e, \rho \in \mathrm{Irr}^u(\mathcal{A}_\lambda)\}/G^\vee \tag{15}$$

and denote its elements by $[\lambda, \rho]$. For fixed λ with $[\lambda] \in \Lambda^e$ we denote by $\tilde{\Lambda}^u_\lambda$ the fiber of $\tilde{\Lambda}^u$ above $[\lambda]$ (with respect to the projection of $\tilde{\Lambda}^u$ to the first factor). We will often simply write $\tilde{\Lambda}$ if we refer to the space of (orbits of) unramified local Langlands data of the quasisplit group $G = G^1$.

The isomorphism classes of pure inner forms G^u are parametrized canonically by $\omega_u \in \mathrm{Irr}({}^L Z_{\mathrm{sc}})$. In the refined version of the local Langlands correspondence where we restrict ourselves to pure inner forms of G, it is therefore more natural to work with pairs (λ, ρ) with $\rho \in \mathrm{Irr}^u(A_\lambda)$, the set of irreducible characters of A_λ which restrict to a multiple of ω_u on ${}^L Z$ (hence there is no need to make choices of the extensions ζ_u in this case).

It is well known [3, Paragraph 6.7] that we have a canonical isomorphism

$$\beta : (G^\vee \times \theta)/\mathrm{Int}(G^\vee) \xrightarrow{\sim} \mathrm{Hom}(X^\theta, \mathbb{C}^\times)/W_0^\theta. \tag{16}$$

Observe that the group $(\Omega^\theta)^*$ of unramified characters on G^{F_u} is exactly the "central subgroup" of the complex torus $T_{\mathbf{v}}(\mathbb{C}) := \mathrm{Hom}(X^\theta, \mathbb{C}^\times)$, i.e., the subgroup of W_0^θ-invariant elements. Here we consider T as the diagonalizable group scheme with character lattice $\mathbb{Z} \times X^\theta$ over the ring $\mathbf{L} = \mathbb{C}[v^{\pm 1}]$ and we use the notation $T_{\mathbf{v}}$ to denote its fiber over $\mathbf{v} \in \mathbb{C}^\times$.

We have natural compatible actions of $X^*_{\mathrm{un}}(G^{F_u}) = (\Omega^\theta)^*$ on the sets Λ and $\tilde{\Lambda}^u$ defined by $\omega[\lambda] = [\omega\lambda]$ and $\omega[\lambda, \rho] = [\omega\lambda, \rho]$ respectively, provided that we choose the extensions ζ_u in a compatible way within each orbit under $X^*_{\mathrm{un}}(G^{F_u})$ (for pure inner forms we do not need to worry about this).

We remark that $W_0\backslash T_{\mathbf{v}}(\mathbb{C})$ can be identified with the maximal spectrum $S^{IM}_{\mathbf{v}}$ of the center $\mathcal{Z}^{IM}_{\mathbf{v}}$ of the Iwahori–Hecke algebra $\mathcal{H}^{IM}_{\mathbf{v}} = \mathcal{H}^{(\mathbb{B},1)}(G^F)$ of the group of points of the k-quasisplit group $G^F = G(k)$. By the Kazhdan–Lusztig correspondence [28] there exists a canonical bijection between the set of central characters $W_0 r_{\mathbf{v}} \in S^{IM}_{\mathbf{v}}$ supporting discrete series representations of $\mathcal{H}^{IM}_{\mathbf{v}}$ and the set of $G^\vee$-orbits of unramified elliptic local Langlands parameters (see [52, Appendix] for the split case; this extends to the quasi-split case using [4, Proposition 6.7] and [61] on the Langlands parameter side, and [39,40,43], and [56] on the Hecke algebra side). This bijection $[\lambda] \to W_0 r_\lambda$ is defined by

$$W_0 r_{\lambda,\mathbf{v}} = \beta\left(G^\vee \cdot \lambda\left(\mathrm{Frob}, \begin{pmatrix} \mathbf{v} & 0 \\ 0 & \mathbf{v}^{-1} \end{pmatrix}\right)\right). \tag{17}$$

This map is equivariant with respect to the natural action of the group $X^*_{\mathrm{un}}(G^{F_u})$ of weakly unramified characters of G^{F_u}.

2.4 Unipotent affine Hecke algebras

According to [40, 1.15, 1.16, 1.17, 1.20] we can decompose for each cuspidal unipotent pair $(\mathbb{P}, \delta)$ of G^u the algebra $\mathcal{H}^{u,\mathfrak{s}}$ of $\mathfrak{s}$-spherical functions on G^u explicitly as a direct sum of mutually isomorphic extended affine Hecke algebras as follows.

Let us use the shorthand notation $N\mathbb{B}$ for $N_G(\mathbb{B})$ etc. Recall that, since Borel subgroups of a connected reductive group are mutually conjugate and self normalizing, the group $\Omega^{\mathbb{P}} = N\mathbb{P}/\mathbb{P}$ is naturally a subgroup of the finite abelian group $\Omega = N\mathbb{B}/\mathbb{B}$ [see (10)]. It is known that the group $\Omega^{\mathbb{P},\theta}$ [see (10)] acts trivially on the set of irreducible unipotent cuspidal representations of $\mathbb{P}^{F_u}$. Even more is true [40]: for every cuspidal unipotent representation (E, δ) of $\mathbb{P}^{F_u}$ there exists an extension $(E, \tilde{\delta})$ of (E, δ) to the normalizer $N\mathbb{P}^{F_u} := N_{G^{F_u}}(\mathbb{P}^{F_u})$ of $\mathbb{P}^{F_u}$ in G^{F_u}. We denote the group $\Omega^{\mathbb{P},\theta}$ by $\Omega^{\mathfrak{s},\theta}$ to stress the invariance of the cuspidal pair $\mathfrak{s} = (\mathbb{P}^{F_u}, \delta)$. One observes that the set of such extensions is a torsor for the group $(\Omega^{\mathfrak{s},\theta})^*$ of irreducible characters of $\Omega^{\mathfrak{s},\theta} = N\mathbb{P}^{F_u}/\mathbb{P}^{F_u}$ by tensoring. Hence the group $X^*_{\mathrm{un}}(G^{F_u}) = (\Omega^\theta)^*$ of weakly unramified characters of G^{F_u} acts transitively on the set of extensions of (E, δ) to $N\mathbb{P}^{F_u}$, and the kernel of this action is equal to the subgroup $(\Omega^\theta/\Omega^{\mathfrak{s},\theta})^*$ of $(\Omega^\theta)^*$ of weakly unramified characters of G^{F_u} which restrict to 1 on $N\mathbb{P}^{F_u}$.

Lusztig showed that the $\mathfrak{s}$-spherical Hecke algebra $\mathcal{H}^{u,\mathfrak{s}}$ is of the form

$$\mathcal{H}^{u,\mathfrak{s}} = \mathcal{H}^{u,\tilde{\mathfrak{s}},a} \rtimes \Omega^{\mathfrak{s},\theta} \tag{18}$$

where $\mathcal{H}^{u,\tilde{\mathfrak{s}},a}$ is an unextended affine Hecke algebra associated with a certain affine Coxeter group $(W_{\mathfrak{s}}, S_{\mathfrak{s}})$ and a parameter function $m_S^{\mathfrak{s}}$, all defined in terms of the pair $\mathfrak{s} = (\mathbb{P}^{F_u}, \delta)$. In particular, they are independent of the chosen extension $\tilde{\mathfrak{s}}$ of $(\mathbb{P}, \delta)$ to $N\mathbb{P}^{F_u}$; for this reason we will often suppress the extension in the notation and write $\mathcal{H}^{u,\mathfrak{s},a}$ instead of $\mathcal{H}^{u,\tilde{\mathfrak{s}},a}$.

In order to define a normalized affine Hecke algebra (in our sense) from these data one needs to choose a distinguished set $S_{\mathfrak{s},0} \subset S_{\mathfrak{s}}$. Although this is not canonically defined, different choices are related via admissible isomorphisms. Let $\Omega_1^{\mathfrak{s},\theta} \subset \Omega^{\mathfrak{s},\theta}$ be the subgroup which acts trivially on $S_{\mathfrak{s}}$ [see (11)]. Then the quotient $\Omega_2^{\mathfrak{s},\theta} = \Omega^{\mathfrak{s},\theta}/\Omega_1^{\mathfrak{s},\theta}$ acts faithfully on $(W_{\mathfrak{s}}, S_{\mathfrak{s}})$ by special affine diagram automorphisms. Lusztig [39, 1.20] showed that $\mathcal{H}^{u,\mathfrak{s}}$ is isomorphic to the tensor product of the group algebra $\mathbb{C}[\Omega_1^{\mathfrak{s},\theta}]$ and the crossed product

$$\mathcal{H}^{u,\tilde{\mathfrak{s}},e} = \mathcal{H}^{u,\tilde{\mathfrak{s}},a} \rtimes \Omega_2^{\mathfrak{s},\theta} \tag{19}$$

which is an extended affine Hecke algebra. Recall from [54, Proposition 2.3] that this information is enough to recover a pair of data (independent of the chosen extension $\tilde{\mathfrak{s}}$ of $\mathfrak{s}$) $(\mathcal{R}^{u,\mathfrak{s}}, m^{u,\mathfrak{s}})$ such that we have an admissible isomorphism $\mathcal{H}^{u,\mathfrak{s},e} \xrightarrow{\sim} \mathcal{H}(\mathcal{R}^{u,\mathfrak{s}}, m^{u,\mathfrak{s}})$ of normalized affine Hecke algebras.

2.4.1 The normalization of the algebras $\mathcal{H}^{u,\mathfrak{s},e}$ Observe that the unit element of $\mathcal{H}^{u,\mathfrak{s}}$ is the function $e^{\mathfrak{s}}$ on G^{F_u} supported on $\mathbb{P}^{F_u}$ defined by

$$e^{\mathfrak{s}}(g) = \mathrm{Vol}(\mathbb{P}^{F_u})^{-1}\chi(g)\delta(g) \tag{20}$$

where χ denotes the characteristic function of $\mathbb{P}^{F_u}$.

Fix an extension $\tilde{\mathfrak{s}}$ of $\mathfrak{s}$ as in the previous paragraph. By (10) the unit element $e^{\mathfrak{s}}$ can be decomposed as a sum of mutually orthogonal idempotents

$$e^{\mathfrak{s}} = \sum_{\lambda \in (\Omega^{\mathfrak{s},\theta})^*} e^{\tilde{\mathfrak{s}}}\lambda \tag{21}$$

where we view $\lambda \in (\Omega^{\mathfrak{s},\theta})^*$ as a linear character of $N\mathbb{P}^{F_u}$ and where

$$e^{\tilde{\mathfrak{s}}}(g) = \mathrm{Vol}(N\mathbb{P}^{F_u})^{-1}\chi_{N\mathbb{P}^{F_u}}(g)\tilde{\delta}(g). \tag{22}$$

By (19) (and the text just above it) we see that the unit element of $\mathcal{H}^{u,\tilde{\mathfrak{s}},e}$ is equal to

$$e^{\tilde{\mathfrak{s}},e} = \sum_{\lambda \in (\Omega_2^{\mathfrak{s},\theta})^*} e^{\tilde{\mathfrak{s}}}\lambda. \tag{23}$$

In particular, the group $(\Omega^{\mathfrak{s},\theta})^*$ acts transitively on the set of idempotents $e^{\tilde{\mathfrak{s}},e}$ obtained by choosing different extensions $\tilde{\mathfrak{s}}$ of $\mathfrak{s}$, and the kernel of this action is the subgroup $(\Omega_2^{\mathfrak{s},\theta})^* \subset (\Omega^{\mathfrak{s},\theta})^*$.

The other canonical basis elements of $\mathcal{H}^{u,\tilde{\mathfrak{s}},e}$ are supported on other double cosets of $N\mathbb{P}^{F_u}$. In particular, the trace τ vanishes on those other basis elements. Hence τ is a multiple of the standard trace of the affine Hecke algebra $\mathcal{H}^{u,\tilde{\mathfrak{s}},e}$, and the normalization factor is of the form

$$d^{\tau,\tilde{\mathfrak{s}},e} := \tau(e^{\tilde{\mathfrak{s}},e}) = |\Omega_1^{\mathfrak{s},\theta}|^{-1}\mathrm{Vol}(\mathbb{P}^{F_u})^{-1}\mathrm{deg}(\delta). \tag{24}$$

The rational number $d^{\tau,\mathfrak{s},e}$ is the evaluation of a Laurent polynomial in the square root $\mathbf{v}$ of the cardinality $\mathbf{q}$ of the residue field $\mathbf{k}$. When we treat $\mathbf{v}$ and $\mathbf{q}$ as an indeterminate we will denote these as v and q respectively. By our normalization of the Haar measure the factor $\mathrm{Vol}(\mathbb{P}^{F_u})$ in the denominator is equal to, up to a power of $\mathbf{v}$, the cardinality of the group of $\mathbf{k}$-points of the reductive group $\overline{\mathbb{P}}$ with Frobenius action F_u. Therefore all factors in $d^{\tau,\tilde{\mathfrak{s}},e}$ are explicitly known rational function in $\mathbf{v}$ (cf. [9, Section 2.9, Section 13.7]). The following property of $d^{\tau,\mathfrak{s},e}$ is very convenient:

Proposition 2.5 *Let $\overline{\mathbb{T}} = \overline{\mathbb{T}}_Z\overline{\mathbb{T}}_S$ denote a maximal F_u-stable, maximally* $\mathbf{k}$*-split torus of $\overline{\mathbb{P}}$, with $\overline{\mathbb{T}}_Z$ the maximal central subtorus. Let V_Z (resp. V_S) denote the rational vector space spanned by the algebraic character lattice L_Z (resp. L_S) of $\overline{\mathbb{T}}_Z$ (resp. $\overline{\mathbb{T}}_S$), and let F_Z (resp. F_S) be the automorphism of L_Z (resp. L_S) induced by F_u. Then we have*

$$d^{\tau,\mathfrak{s},e} = \pm|\Omega_1^{\mathfrak{s},\theta}|^{-1}det_{V_Z}(\mathbf{v}Id_{V_Z} - \mathbf{v}^{-1}F_Z)^{-1}\prod_{i=1}^{l}(\mathbf{v}^{d_i} - \epsilon_i\mathbf{v}^{-d_i})^{-1}\mathbf{v}^a deg_{\mathbf{v}}(\delta) \tag{25}$$

where l is the semisimple rank of $\overline{\mathbb{P}}$ over $\mathbb{K}$, d_i are the primitive degrees of the Weyl group invariants of the semisimple part of $\overline{\mathbb{P}}$, the ϵ_i are the eigenvalues of F_S acting on the co-invariant ring with respect to the Weyl group action on V_S (certain roots of *unity, see* [9, Section 2.9]*), and where $a \in \mathbb{Z}$ is such that $f(\mathbf{v}) = \mathbf{v}^a deg_{\mathbf{v}}(\delta)$ satisfies $f(\mathbf{v}^{-1}) = \pm f(\mathbf{v})$. At $v = 1$, $d^{\tau,\mathfrak{s},e}$ has a pole of order equal to the split rank r_Z of $\overline{\mathbb{T}}_Z$, and satisfies $d^{\tau,\mathfrak{s},e}(v) = (-1)^{r_Z} d^{\tau,\mathfrak{s},e}(v^{-1})$.*

Proof For G containing a k-split torus of positive dimension, then this is an easy case-by-case verification using [9, Section 2.9, Section 13.7]. The anisotropic case is easy by the results stated in 2.2.3. □

As a consequence, with our normalization of Haar measures, the normalization constant $d^{\tau,\mathfrak{s},e}$ of a unipotent affine Hecke algebras $\mathcal{H}^{u,\mathfrak{s},e}$ satisfies the condition of [54, 3.1.2] and, at $v = 1$, has a pole of order equal to the rank of $\mathcal{H}^{u,\mathfrak{s},e}$. Hence by Theorem [54, Theorem 4.8](iii), in our normalization of Haar measures all formal degrees of the discrete series representations of the unipotent affine Hecke algebras, and thus of all unipotent discrete series representations, are symmetric with regards to $v \to v^{-1}$, and regular and nonzero at $v = 1$. This is convenient, since it implies that we never need to be concerned about the factors v^N or of $(v - v^{-1})^M$ of the formal degree of a unipotent discrete series: With our normalizations these factors do not appear in $\mathrm{fdeg}(\pi)$.

Definition 2.6 Let $\mathbf{K}^\times$ be the field of rational fundtions in v. Recall the notion of a normalized affine Hecke algebra [54, Definition 2.13]. Given our normalization of the traces, we see from [54, Theorem 4.8] and Theorem 2.4 that the formal degree $\mathrm{fdeg}(\pi)$ of a discrete series representation π of a unipotent Hecke algebra has a unique representation $\mathrm{fdeg}(\pi) = \lambda \mathrm{fdeg}(\pi)_q \in \mathbf{K}^\times$ where $\lambda \in \mathbb{Q}_+$, and $\mathrm{fdeg}(\pi)_q$ is a q-rational number (by which we mean a fraction of products of q-integers $[n]_q := \frac{\mathbf{v}^n - \mathbf{v}^{-n}}{\mathbf{v} - \mathbf{v}^{-1}}$ with $n \in \mathbb{N}$). We call $\mathrm{fdeg}(\pi)_q$ the q-rational factor of $\mathrm{fdeg}(\pi)$.

Corollary 2.7 *For each $\omega \in \Omega/(1-\theta)\Omega$ (with representative $u \in N\mathbb{B}^{F_u}$ as before) and each cuspidal unipotent pair $\mathfrak{s}$ of G^u, the pair $(\mathcal{H}^{u,\mathfrak{s},e}, d^{\tau,\mathfrak{s},e})$ is a normalized affine Hecke algebra in the sense of Definition* [54, Definition 3.1]. *The group $(\Omega_2^{\mathfrak{s},\theta})^*$ acts naturally on the algebra $\mathcal{H}^{u,\mathfrak{s},e} = \mathcal{H}^{u,\mathfrak{s},a} \rtimes \Omega_2^{\mathfrak{s},\theta}$ by means of essentially strict automorphisms (cf.* [54, paragraphs 2.1.7 and 3.3.3]*) (in particular, this action induces spectral measure preserving automorphisms on the tempered spectrum of $(\mathcal{H}^{u,\mathfrak{s},e}, d^{\tau,\mathfrak{s},e})$). The abelian group $(\Omega^{\mathfrak{s},\theta})^*$ acts similarly by essentially strict automorphisms on $\mathcal{H}^{u,\mathfrak{s}} \xrightarrow{\sim} \mathcal{H}^{u,\mathfrak{s},a} \rtimes \Omega^{\mathfrak{s},\theta} \approx \mathcal{H}^{u,\mathfrak{s},e} \otimes \mathbb{C}[\Omega_1^{\mathfrak{s},\theta}]$. This action is transitive on the set of direct summands of the form $(\mathcal{H}^{u,\tilde{\mathfrak{s}},e}, d^{\tau,\mathfrak{s},e})$ where $\tilde{\mathfrak{s}}$ runs over the set of extensions of $\mathfrak{s}$ to $N\mathbb{P}^{F_u}$. The subgroup $(\Omega_2^{\mathfrak{s},\theta})^* \subset (\Omega^{\mathfrak{s},\theta})^*$ is the kernel of the induced action on the set of these direct summands.*

Recall that $N_{G^u(k)}(\mathbb{B}^{F_u})/\mathbb{B}^{F_u} \xrightarrow{\sim} \Omega_C^\theta$ by (10). In particular this group acts naturally on the set of F_u-stable standard cuspidal parahoric subgroups of G^u. This action extends naturally to an action on the set of equivalence classes of standard cuspidal unipotent pairs $\mathfrak{s} = (\mathbb{P}, \delta)$ by $\omega \cdot (\mathbb{P}, \delta) = ({}^\omega\mathbb{P}, {}^\omega\delta)$; as was remarked before, the isotropy group of $\mathfrak{s} = (\mathbb{P}, \delta)$ is the same as that of its first component $\mathbb{P}$. If $\omega \cdot \mathfrak{s}_1 = \mathfrak{s}_2$, then conjugation by $\omega \in \Omega_C^\theta$ gives rise to an isomorphism $\phi_\omega : \mathcal{H}^{u,\mathfrak{s}_1} \xrightarrow{\sim} \mathcal{H}^{u,\mathfrak{s}_2}$ which maps the various normalized extended affine Hecke algebra summands of the form $(\mathcal{H}^{u,\tilde{\mathfrak{s}}_1,e}, d^{\tau,\mathfrak{s}_1,e})$ in $\mathcal{H}^{u,\mathfrak{s}_1}$ to corresponding normalized extended affine Hecke algebra summands of $\mathcal{H}^{u,\mathfrak{s}_2}$ by essentially strict isomorphisms.

Given an orbit $\mathcal{O}$ of standard cuspidal unipotent pairs $\mathfrak{s}$ of G^u for action of the group Ω_C^θ, one can form the crossed product algebra

$$\mathcal{H}^{u,\mathcal{O}} = (\bigoplus_{\mathfrak{s}\in\mathcal{O}} \mathcal{H}^{u,\mathfrak{s}}) \rtimes \Omega_C^\theta. \tag{26}$$

Then $\mathcal{H}^{u,\mathcal{O}}$ is Morita equivalent to the direct sum $\mathcal{H}^{u,\mathfrak{s}} \rtimes \Omega^{\mathfrak{s},\theta}$. If (V,π) is an object of $R(G^u(k))_{\text{uni}}$, let $V_{\mathfrak{s}}$ denote the $\mathfrak{s}$-isotypical component of $V|_{\mathbb{P}^{F_u}}$ (where $\mathfrak{s} = (\mathbb{P},\delta)$), and put $V^{\mathfrak{s}} = \mathrm{Hom}_{\mathbb{P}}(\delta, V|_{\mathbb{P}})$. Then

$$V^{u,\mathcal{O}} = \oplus_{\omega\in\Omega^\theta/\Omega^{\mathfrak{s},\theta}}(\pi(\omega)V^{\mathfrak{s}}) = \oplus_{\mathfrak{s}'\in\mathcal{O}} V^{\mathfrak{s}'} = \oplus_{\mathfrak{s}'\in\mathcal{O}} e^{\mathfrak{s}'} V^{u,\mathcal{O}} \tag{27}$$

is a representation of $\mathcal{H}^{u,\mathcal{O}}$ (see also paragraph 2.2.4). Here $e^{\mathfrak{s}'}$ denotes the unit element of $\mathcal{H}^{u,\mathfrak{s}'}$.

The Pontryagin dual $X^*_{\text{un}}(G^{F_u}) = (\Omega_C^\theta)^*$ of Ω_C^θ acts in a natural way on the algebra $\mathcal{H}^{u,\mathcal{O}}$ by automorphisms as follows. If $\chi \in (\Omega_C^\theta)^*$, then the corresponding automorphism α_χ acts as the identity on the subalgebra $\oplus\mathcal{H}^{u,\mathfrak{s}}$, while $\alpha_\chi(\omega) = \chi(\omega)\omega$. If $\chi \in (\Omega_C^\theta/\Omega^{\mathfrak{s},\theta})^*$ (i.e., $\chi|_{\Omega^{\mathfrak{s},\theta}} = 1$), then α_χ is the inner automorphism obtained by conjugation with $\sum_{\omega\in\Omega^\theta/\Omega^{\mathfrak{s},\theta}} \chi(\omega)e^{\omega\mathfrak{s}}$. In particular the subgroup $(\Omega_C^\theta/\Omega^{\mathfrak{s},\theta})^*$ of $X^*_{\text{un}}(G^{F_u})$ acts trivially on the set of irreducible representations of $\mathcal{H}^{u,\mathcal{O}}$.

The results of this paragraph can be summarized as follows:

Theorem 2.8 *Let G be a connected absolutely quasisimple K-split, k-quasisplit linear algebraic group. Consider the cartesian product $R(G)_{uni} := \prod_u R(G^{F_u})_{uni}$, where $R(G^{F_u})_{uni}$ denotes the category of unipotent representations of G^{F_u}, and where the product is taken over a complete set of representatives of classes of pure inner k-forms $[u] \in H^1(F,G)$ of G. Let $\mathcal{M}$ be the category of modules over the direct sum of algebras $\mathcal{H}_{uni} := \oplus_{u,\mathcal{O}}\mathcal{H}^{u,\mathcal{O}}$, where the direct sum is taken over the a complete set of representatives of classes of inner k-forms $[u] \in H^1(F,G)$ of G and $X^*_{un}(G)$-orbits $\mathcal{O}$ of standard cuspidal unipotent pairs $\mathfrak{s}$ of G^u. Consider the functor $U : R(G)_{uni} \to \mathcal{M}$ defined by sending V to $\oplus_{u,\mathcal{O}} V^{u,\mathcal{O}}$.*

- *(i) The functor U is an equivalence of categories.*
- *(ii) For each orbit $\mathcal{O}$ of standard cuspidal unipotent pairs of G^u and each $\mathfrak{s} \in \mathcal{O}$, the irreducible spectrum of $\mathcal{H}^{u,\mathcal{O}}$ is in canonical Morita bijection with the irreducible spectrum of $\mathcal{H}^{u,\mathfrak{s}}$. In turn this equals the disjoint union of the irreducible spectra of the direct summands $\mathcal{H}^{u,\tilde{\mathfrak{s}},e}$ of $\mathcal{H}^{u,\mathfrak{s}}$, where $\tilde{\mathfrak{s}}$ runs over the collection of distinct extensions of $\mathfrak{s}$ to $N\mathbb{P}^{F_u}$ (this collection is a $(\Omega_1^{\mathfrak{s},\theta})^*$-torsor). We define the tempered spectrum and spectral measure of $\mathcal{H}^{u,\mathcal{O}}$ via these canonical bijections.*
- *(iii) The bijection $[U]$ that U induces on the irreducible spectrum restricts to a homeomorphism $[U]^{temp}$ from the disjoint union of the tempered unipotent spectra of the classes of pure inner forms G^u of G to the disjoint union of the tempered spectra of the various $\mathcal{H}^{u,\mathcal{O}}$.*

(iv) *The push forward of the union of the Plancherel measures of the various G^{F_u} under the bijection $[U^{temp}]$ is the union of the spectral measures of the various $\mathcal{H}^{u,\mathcal{O}}$.*
(v) *For each $\mathfrak{s} \in \mathcal{O}$ the action of $X^*_{un}(G) = (\Omega^\theta_C)^*$ on the irreducible spectrum of $\mathcal{H}^{u,\mathcal{O}}$ is trivial on the subgroup $(\Omega^\theta_C/\Omega^{\mathfrak{s},\theta})^*$. The quotient $(\Omega^{\mathfrak{s},\theta})^*$ of $(\Omega^\theta_C)^*$ acts on the spectrum of $\mathcal{H}^{u,\mathcal{O}}$ via the canonical Morita bijection of this set with the spectrum of $\mathcal{H}^{u,\mathfrak{s}}$ (which is naturally a $(\Omega^{\mathfrak{s},\theta})^*$-set by Corollary 2.7).*
(vi) *The group $X^*_{un}(G) = (\Omega^\theta_C)^*$ acts on $\mathcal{H}^{u,\mathcal{O}}$ via spectral automorphisms. In particular, this action induces a measure preserving action on the tempered spectrum of $\mathcal{H}^{u,\mathcal{O}}$. Moreover, via the bijection $[U^{temp}]$ this action corresponds with the natural action of $X^*_{un}(G)$ on $R(G)_{uni}$ by taking tensor products.*

3 The spectral transfer category of unipotent Hecke algebras

3.1 Spectral transfer morphisms

Recall the notion of a spectral transfer morphism (STM) $\phi : \mathcal{H}_1 \rightsquigarrow \mathcal{H}_2$ between two normalized affine Hecke algebras as introduced in [54, Definition 5.1, Definition 5.9]. In this section we will classify the STMs between unipotent affine Hecke algebras (which will be referred to as "unipotent STMs").

3.1.1 Restriction of STMs Let $(\mathcal{H}, \tau)$ denote a normalized affine Hecke algebra, and let L denote a generic residual coset $L \subset T$ for $\mathcal{H}$. Then there exists a unique "parabolic subsystem" $R_P \subset R_0$ such that L can be written in the form $L = rT^P$ with $r \in L \cap T_P$. After moving L with a suitable Weyl group element $w \in W_0$, we may assume that R_P is standard and associated with a subset $P \subset F_0$. To this subset we may associate a subalgebra $\mathcal{H}^P$ ("a standard Levi subalgebra") and its semisimple quotient algebra $\mathcal{H}_P$ whose associated algebraic torus is the subtorus $T_P \subset T$ (cf. [52]). In this situation $\{r\} \subset T_P$ is a residual point for $\mathcal{H}_P$.

Definition 3.1 We will normalize the affine Hecke algebra $\mathcal{H}^P$ by the trace τ^P defined by $\tau^P(1) = \tau(1)$. We normalize $\mathcal{H}_P$ by the trace τ_P defined by the property $\tau_P(1) = (v - v^{-1})^{\mathrm{rk}(R_0)-\mathrm{rk}(R_P)}\tau(1)$

Suppose that $\phi : (\mathcal{H}', \tau') \rightsquigarrow (\mathcal{H}, \tau)$ is a strict STM which is represented by $\phi_T : T' \to L_n$ with $L = rT^P$ a residual coset. By modifying the representing map ϕ_T appropriately, we may assume that $rK^n_L = \phi_T(e)$ and such that $D\phi_T(\mathfrak{t}') = \mathfrak{t}^P$ for some subset $P \subset F_m$. It follows easily from Corollary [54, Corollary 5.7] and Corollary [54, Corollary 5.8] that for any inclusion $P \subset Q \subset F_m$, after possibly modifying the representing morphism ϕ_T by a Weyl group element again, the inverse image $\phi_T^{-1}(K^n_L(L \cap T_Q)/K^n_L) \subset T'$ is a *subgroup* whose identity component is a subtorus of T' with as Lie algebra a subspace of $\mathfrak{t}' := \mathrm{Lie}(T')$ of the form $\mathfrak{t}_{Q'}$ for some standard parabolic subsystem $Q' \subset F'_m$. Indeed, in Corollary [54, Corollary 5.7] we saw that $D\phi_T$ induces a bijective correspondence between parabolic subsystems $R_{Q'}$ of R'_m and parabolic subsystems R_Q of R_m containing R_P. By modifying ϕ_T with an appropriate Weyl group element $w' \in W(R'_m)$, we may assume that $R_{Q'} = (D\phi_T)^{-1}(R_Q)$ is standard, associated to a subset $Q' \subset F'_m$. By Definition 3.1 and the definition of an STM it is easy to see that in this context, ϕ_T also defines an STM $\phi^Q : \mathcal{H}'^{Q'} \rightsquigarrow \mathcal{H}^Q$, and that the restriction $\phi_{T,Q}$ of ϕ_T to $T'_{Q'} \subset T'$ defines an STM $\phi_Q : \mathcal{H}'_{Q'} \rightsquigarrow \mathcal{H}_Q$ (Recall that $\dot{K}_L^n = N_{W_P}(L)$, and so L_n is also the image of ϕ^Q. If T_Q^P denotes the identity component of $T_Q \subset T^P$, then $L_Q := rT_Q^P \subset T_Q$ is a residual coset of T_Q. Thus $L_{Q,n} = L_Q/K_L^n \cap T_Q^P$ so that $L_{Q,n} \subset L_n$. Hence $L_{Q,n}$ is the image of the restriction of ϕ_Q).

Definition 3.2 We call ϕ_Q the restriction of ϕ to $\mathcal{H}'_{Q'}$, and we say that ϕ is *induced* from ϕ_Q. In particular, ϕ is induced by the rank 0 STM $\phi_P : \mathbf{L} \rightsquigarrow \mathcal{H}_P$.

3.1.2 Induction of unipotent STMs By the above, every STM is induced from a rank 0 transfer map. The converse is clearly not true: not every rank 0 STM of the form $\psi : \mathcal{H}'' = \mathbf{L} \to \mathcal{H}_P$ is the restriction of an STM $\Psi : \mathcal{H}' \rightsquigarrow \mathcal{H}$. Indeed, if $\mathrm{Im}(\psi) = r$ (a generic residual point of the subtorus $T_P \subset T$), then we should have $\mathrm{Im}(\Psi) = L = rT^P$. But the spectral measure ν_{Pl} on a component $\mathfrak{S}_{(P,\delta)}$, where δ is a discrete series representation of $\mathcal{H}_P$ with central character $W_P r$, is given (up to a rational constant depending on δ) by the restriction of the regularisation $\mu^{(L)}|_{L^{\mathrm{temp}}}$ of the μ-function to $S_{(P,\delta)} = W_0\backslash W_0 L^{\mathrm{temp}}$ (cf. Theorem [54, Theorem 4.13]). This regularisation does in general not behave like a μ-function of an affine Hecke algebra, unless for every restricted root of $R_0\backslash R_P$ to L the appropriate cancellations occur. However, as we will see in 3.1.3., if $\mathcal{H} = \mathcal{H}^{IM}(G)$ for a quasisplit almost simple algebraic groups G, and $\mathcal{H}''$ is the normalized Hecke algebra for a maximal cuspidal unipotent pair $(\mathbb{P}, \sigma)$ of an inner form of the standard Levi subgroup of G associated to $P \subset F_0$, then ψ *will* be the restriction of an STM $\Psi : \mathcal{H}' \rightsquigarrow \mathcal{H}$.

3.1.3 Induction and cuspidality of unipotent STMs This brings us to an informal discussion of the heuristic ideas and surprising facts behind the notion of STMs between unipotent affine Hecke algebras (with their canonical normalizations as in 2.4.1). We refer to such STMs as "*unipotent STMs*". Let G be an absolutely quasisimple unramified group over k, and let G^u denote a k-group in the same inner class. We fix a maximal K-split torus $S \subset G$ defined over k. Let the automorphism induced by the action of the Frobenius F of G on the character lattice of S be denoted by θ. We will assume that $u = \dot{\omega} \in N\mathbb{B}$ is a representative of an element $\bar{\omega} \in \Omega/(1-\theta)\Omega$ (as in paragraph 2.2.4). We choose a minimal F-stable parabolic subgroup $A_0 \subset G$.

These data give rise to the "local index" of G^F, a (possibly twisted) affine Dynkin diagram which contains a hyperspecial node, whose underlying finite root system is the restricted root system of G^F with respect to the k-split center of A_0^F (again, we apologize for denoting the restricted roots of $G(k)$ as "coroots"). We can now "untwist" the affine diagram by doubling some of the restricted roots of $G(k)$; the resulting root system is denoted by $R_0^\vee$. We have thus associated a based root datum $\mathcal{R} := (R_0, X, R_0^\vee, Y, F_0^\vee)$ such that the "untwisted" local index of $G(k)$ equals $(R_0^\vee)^{(1)}$, and u acts on this affine diagram via the action of ω as a special affine diagram automorphism. Notice that u acts naturally on the root system R_0 by an element $w_u \in W_0(R_0)$. The local index comes equipped with integers $m_S(a_i)$ attached to the nodes a_i, which we transfer unaltered to the nodes of the untwisted diagram. This is the arithmetic diagram $\Sigma_a(\mathcal{R}, m)$ of [54, Subsection 2.3] associated to $\mathcal{H}^{IM}$. The associated spectral diagram $\Sigma_s(\mathcal{R}, m)$ is an untwisted affine Dynkin diagram for the affine root system $\mathcal{R}^m = R_m^{(1)}$. Let T be the complex algebraic torus with character lattice X.

The first remarkable fact is that for a cuspidal unipotent representation σ_u of G^u, its formal degree equals (up to a rational constant) the formal degree of an Iwahori-spherical unipotent discrete series representation δ of G, and the central character $W_0 r$ of the corresponding discrete series representation δ_σ of the Iwahori–Hecke algebra $\mathcal{H}^{IM} := \mathcal{H}^{IM}(G, \mathbb{B})$ of G is uniquely determined by this formal degree, up to the action of $X^*_{\text{un}}(G)$. Here r is a generic residual point. (This is the rank 0 case of Theorem 3.4 that we already mentioned above). Let us agree to call a residual point r of $\mathcal{H}^{IM}$ *cuspidal* if the q-rational factor of the residue $\mu^{IM,(\{r\})}(r)$ equals the q-rational factor of the formal degree of a cuspidal unipotent representation σ^u for an inner form G^u of G as above.

We claim that this is also true if we replace G^u by a proper Levi subgroup $M^u = C_{G^u}(S^u)^0$ of G^u (with $S^u \subset G^u$ the k-split part of the connected center of M^u) which carries a cuspidal unipotent representation σ^u_M, in the following sense. We may assume that $S^u \subset S$, the subtorus of S defined by the vanishing of the K-roots of M^u. Then $S^u \subset S$ also gives rise to a k-Levi subgroup $M = C_G(S^u)^0$ of G with connected center S^u. Observe that M is k-quasisplit itself, and that M^u is an inner form of M (since $\dot{\omega} \in M$).

Let $R^\vee_M \subset R^\vee_0$ denote the set of (restricted) K-roots of M^u. Since σ^u is unipotent, it factors through a cuspidal unipotent representation σ^u_M of the quotient $M^u_{ssa} := M^u/S^u$. (This quotient consists of an almost product of a semisimple group and a central anisotropic torus.) Then σ^u_M first of all uniquely determines an orbit of cuspidal residual points $W_M r_M \subset T_M$ of $\mathcal{H}^{IM}_M$, up to the action of the finite subgroup of W_M-invariant characters Ω^*_M of $X_M/\mathbb{Z}R_M$ of T_M (which contains the group $K_M := T_M \cap T^M$). This should still be true if the rank 0 case of Theorem 3.4 holds, even though $M_{ssa} := M/S^u$ is not absolutely quasisimple in general. Namely, all but at most one of the absolutely quasisimple almost factors of M_{ssa} are of type A, and these type A factors admit just one (up to twisting by weakly unramified characters) residual point. The residual point r_M is thus the image of the representing map ϕ_T for a unique cuspidal unipotent STM $\phi_M : \mathbf{L} \rightsquigarrow \mathcal{H}^{IM}(M_{ssa})$, where $\mathcal{H}^{IM}(M_{ssa})$ denotes the Iwahori–Matsumoto Hecke algebra of M_{ssa} with respect to the Iwahori subgroup $M \cap \mathbb{B}/(S^u \cap \mathbb{B})$ of M_{ssa}, up to the action of Ω^*_M.

In particular $W_M r_M$ gives rise to a maximal finite type subdiagram $J_{M,r_M} \subset \Sigma_s(\mathcal{R}_M, m_M)$ (the spectral diagram of M_{ssa}, defined similarly as we did for G in the text above). Namely, J_{M,r_M} is determined by choosing r_M appropriately inside the orbit $W_M r_M$, then $r_M = s_M c_M \in T_{M,u} T_{M,v}$ with s_M defining a vertex of every component of $\Sigma_s(\mathcal{R}_M, m_M)$. To obtain J_{M,r_M}, one needs to strike out these nodes from $\Sigma_s(\mathcal{R}_M, m_M)$. *In all cases, a subset of such type J_M fits in a unique way as an excellent* (cf. [40]) *subset of* $\Sigma_s(\mathcal{R}, m)$. Since $T^M \subset T$ is a maximal subtorus on which the dual affine roots in J_M are constant, we see that the pair (r_M, T^M) is uniquely determined from just the type of M_{ssa} and the q-rational factor of the unipotent degree of σ_M^u, up to the action of $W(R_0)$ and the group Ω_M^*. In particular it is determined by the inertial class of the cuspidal pair (M^u, σ^u).

The cuspidal pair (M^u, σ_M^u) is associated to a unique "extended type" $\mathfrak{s} := (N\mathbb{P}^{F_u}, \tilde{\delta})$ in the sense of [51] (also see paragraph 2.2.4), where $\mathbb{P} \subset G$ is an F_u-stable parahoric subgroup such that $\mathbb{P}^{F_u} \cap M^{F_u}$ is a *maximal* parahoric of M^{F_u}, and such that the set of affine roots associated to the parahoric subgroup $\mathbb{P}$ has a basis given by a proper ω-invariant subset $\Sigma_a(\mathcal{R}, m)$. The Plancherel measure on the set of tempered representations which belong to the unipotent Bernstein component whose cuspidal support is the inertial equivalence class of the cuspidal pair (M^u, σ^u) is given by the Plancherel measure of the normalized unipotent affine Hecke algebra $\mathcal{H}^{u,\tilde{\mathfrak{s}},e}$ (cf. e.g., [8,21,50,51,60]).

Let us now move M to its standard position, so that $R_M^\vee$ is replaced by a standard parabolic subsystem $R_Q^\vee$ of roots associated to a subset $Q \subset F_0$. This corresponds to a standard Levi subgroup $G^Q = C_G(S^u)$ of G which is conjugate to M. Suppose that $\widetilde{\sigma^Q}$ is an Iwahori spherical representation of G^Q which is tempered and L^2 modulo the center of G^Q. The corresponding tempered representation π^Q of $\mathcal{H}_Q^{IM}$ is then the form $\pi^Q = (\pi_Q)_t$ for some *Iwahori spherical* discrete series representation π_Q of $\mathcal{H}_Q^{IM}$ and some $t \in T_u^Q$. Assume now that the central character of π_Q is a *cuspidal* residual point of T_Q. This means by definition that there also exists a cuspidal unipotent representation σ_M^u of some Levi subgroup M_{ss}^u of some inner form G^u as above, whose formal degree has the same q-rational factor as that of $\widetilde{\sigma_Q}$ (or equivalently of π_Q). The second important heuristic ingredient we now apply is the general expectation that the q-rational factor of the formal degree of the members of a discrete series unipotent L-packet should be the equal for all members of the packet [60]. It then follows from the above uniqueness assertions that σ_Q and a twist σ_Q^u of σ_M^u (i.e., σ_Q^u is obtained from σ_M^u by a pull-back via a k-isomorphism between G_Q^u and M^u) must belong to the same L-packet of G_Q.

Recall from paragraph [54, 4.2.5] how the Plancherel measure $\nu_{Pl}|_{\mathfrak{S}_{(P,\sigma)}}$ on the component $\mathfrak{S}_{(P,\sigma)}$ of the tempered spectrum of $\mathcal{H}^{IM}$ associated to a discrete series representation σ of $\mathcal{H}_Q^{IM}$ with central character $W_Q r_Q$ is expressed in terms of the residue $\mu^{IM,(L)}$ where $L = r_Q T^Q$ (see paragraph [54, 4.2.5]). This implies that if discrete series representations σ_1, σ_2 of $\mathcal{H}_P$ are associated to the same central character $W_P r \in W_P \backslash T_P$, then the components $\mathfrak{S}_{(P,\sigma_i)}$ $(i = 1, 2)$ are related to each other by a Plancherel measure preserving (up to a rational constant) correspondence as in Theorem [54, Theorem 6.1].

The third heuristic idea is that *such a correspondence should exist for Plancherel measures on the tempered components determined by any two discrete series induction data* $(G_Q, \widetilde{\sigma_Q})$ *and* (G_Q^u, σ_Q^u) *whenever* $\widetilde{\sigma_Q}$ *and* σ_Q^u *belong to the same L-packet*. But for the latter cuspidal unipotent pair, this Plancherel measure is computed as the most continuous part of the tempered spectrum of the normalized unipotent affine Hecke algebra $\mathcal{H}^{u,\tilde{\mathfrak{s}},e}$. On the other hand, for the first pair it was already discussed above that the Plancherel measure can be computed essentially as the residue measure of the μ-function of $\mathcal{H}^{IM}$ with respect to the tempered residual coset $L^{\text{temp}} = r_Q T_u^Q$. Thus these ideas suggest the existence of a *unique* STM $\mathcal{H}^{u,\tilde{\mathfrak{s}},e} \rightsquigarrow \mathcal{H}^{IM}$ represented by a morphism ϕ_T with image $L = r_Q T^Q$ (or more precisely, the finite quotient $L_n = L/K_L^n$ of L, where $K_L^n \subset K_Q := T^Q \cap T_Q$ is the subgroup of elements whose action on L can be represented by an element of $W(R_Q)$ [see [54, Paragraph 5.1.1]]) associated to the cuspidal pair (M^u, δ^u) (or equivalently, to the extended type $\tilde{\mathfrak{s}}$).

Recall from Proposition [54, Proposition 5.6] that any STM $\mathcal{H}' \rightsquigarrow \mathcal{H}$ represented by an affine morphism $\phi_T : T' \to T$ with image $L = rT^Q$, the subtorus $T^Q \subset T$ is W_0-conjugacte to a subtorus $T^J \subset T$ which is defined as above by an excellent subset J of the spectral diagram of $\mathcal{H}$. Therefore it is clear that if there exists a unipotent STM $\phi : \mathcal{H}^{u,\tilde{\mathfrak{s}},e} \rightsquigarrow \mathcal{H}^{IM}$ as expected by the above discussion, then its image must be uniquely determined by the type of M_{ssa} in combination with the q-rational factor of the formal degree of σ_M^u, up to the action of $X_{\mathrm{un}}^*(G)$. By Proposition [54, Proposition 7.13] we see that ϕ itself is therefore determined up to the action of $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{H}^{u,\tilde{\mathfrak{s}},e})^{\mathrm{op}} \times X_{\mathrm{un}}^*(G)$.

Any unipotent STM $\Phi : \mathcal{H}^{u,\tilde{\mathfrak{s}},e} \rightsquigarrow \mathcal{H}^{IM}$ is induced from a cuspidal unipotent STM $\phi : \mathbf{L} \rightsquigarrow \mathcal{H}_Q^{IM}$ which is uniquely determined modulo the action of K_Q/K_L^n (with $K_L^n \subset K_Q$ as above, hence the image of a representing morphism ϕ_T for Φ is L/K_L^n), this is obvious. But by our discussion above we expect that: *Conversely, any cuspidal unipotent STM* $\phi : (\mathbf{L}, \tau_0) =: \mathcal{H}_0 \rightsquigarrow \mathcal{H}_Q^{IM}$ *for the quotient* $G_Q = G^Q/Z_s(G^Q)^0$ *(where* $Z_s(G^Q)^0$ *is the connected k-split center) of a standard Levi subgroup* G^Q *can be induced to yield a unique spectral transfer morphism* $\Phi : \mathcal{H}^{u,\tilde{\mathfrak{s}},e} \rightsquigarrow \mathcal{H}^{IM}$*. Up to the action of* $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{H}^{u,\tilde{\mathfrak{s}},e})^{\mathrm{op}} \times X_{\mathrm{un}}^*(G)$, Φ *is completely determined by the type of* G_Q *and the q-rational factor of the degree* $\tau_0(1)$ *of* $\mathcal{H}_0$.

In the above arguments two important aspects of cuspidal residual points played a role. The first is that they can be defined by the property that the associated residue degree $\mu^{IM,(\{r\})}(r)$ has the same q-rational factor as that of the formal degree of a cuspidal unipotent representation of some inner form of G. The second is that a cuspidal residual point r_Q of a semisimple quotient Hecke algebra $\mathcal{H}_Q$ of $\mathcal{H}^{IM}$ (where cuspidal means now that there exists a Levi subgroup M^u of an inner form G^u of G which carries a cuspidal unipotent representation σ^u and which is isomorphic to an inner form of G_Q, such that the q-rational factor of the formal degree of σ^u is equal to that of the residue of μ_Q^{IM} at r_Q) is always the restriction of an STM $\mathcal{H}^{u,\tilde{s},e} \rightsquigarrow \mathcal{H}^{IM}$, for any inclusion of $(\mathcal{R}_Q, m_Q)$ as a standard parabolic subsystem of the based root datum $(\mathcal{R}, m)$ (with parameter funtion) of $\mathcal{H}^{IM}(G) = \mathcal{H}(\mathcal{R}, m)$.

A priori the second property seems much more restrictive (except for the "final" exceptional groups E_8, F_4 and G_2), but miraculously these properties lead to the same notion of cuspidality. The essential uniqueness part of Theorem 3.4 reduces to the rank 0 (or cuspidal) case in this way. The cuspidal case is done by direct inspection for the exceptional groups (most of the required results are in [21,59,60]). For the classical groups, the cuspidal case is treated in [19].

Of course the arguments above are only heuristic, but they tell us precisely where we should expect STMs, how these should be defined by induction from cuspidal ones, and what is necessary to check in order to prove that these maps really are STMs (thus providing a Proof of Theorem 3.4). *In the remainder of this paper we will prove that indeed, any unipotent cuspidal pair* $(\mathbb{P}^u, \sigma)$ *of an inner form* G^u *gives rise in this way* *to an essentially unique STM* $\mathcal{H}^{u,\tilde{s},e} \rightsquigarrow \mathcal{H}^{IM}$, thereby proving Theorem 3.4 in full generality.

For exceptional groups the required verifications that induction of cuspidal STMs from Levi subgroups always give rise to STMs is based on the notion of a "transfer map diagram". This notion is defined and discussed in paragraph 3.1.4. One can also study more generally the STMs between two unipotent affine Hecke algebras, not just the ones with $\mathcal{H}^{IM}$ as a target. This is interesting in itself, since in several cases the "unipotent spectral transfer category" is generated by very simple building blocks of this kind. Indeed, this is how we show the existence of STMs induced from cuspidal ones in the classical cases.

3.1.4 The transfer map diagram of a unipotent STM Such an expected unipotent STM $\Phi : \mathcal{H}^{u,\tilde{\mathfrak{s}},e} \rightsquigarrow \mathcal{H}^{IM}$ is determined (up to the action of $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{H}^{u,\tilde{\mathfrak{s}},e})^{\mathrm{op}}$) by the image $r_Q = s_Q c_Q$ of $\phi_{T,Q}$, a cuspidal generic residual point for the Iwahori–Matsumoto Hecke algebra $\mathcal{H}^{IM}_Q$ of the quasisplit Levi subgroup G_Q of G. We can choose the unitary part $s_Q = s(e_Q) \in T_{Q,u}$ such that it corresponds to a vertex $e_Q \in C^{Q,\vee}$, the fundamental alcove for dual affine Weyl group $(W_Q)^\vee_{m_Q}$ associated to $(\mathcal{R}_Q, m_Q)$. Let v_Q be the set of corresponding nodes of the spectral diagram $\Sigma_s(\mathcal{R}_Q, m_Q)$, and put J_Q for the finite type Dynkin diagram that is the complement of v_Q of $\Sigma_s(\mathcal{R}_Q, m_Q)$. Let $\mathcal{R}'$ denote the root datum underlying $\mathcal{H}^{u,\tilde{\mathfrak{s}},e}$, with multiplicity function m'.

A node v_i of the complement J_Q is weighted with the weight $w_i := Da_i^\vee(c_Q)$ of the gradient $Da_i^\vee$ of the corresponding dual affine root $a_i^\vee$ (this value is a power of q). We may put c_Q in a dominant position with respect to the roots $Da_i^\vee$ where i runs over the nodes of J_Q. This is essentially the weighted Dynkin diagram of a linear generic residual point (in the sense of [56], but obviously restricted in our context of the fixed line in the parameter space defined by $m_R^\vee$) for the finite type root system defined by J_Q with the parameters $m_Q^\vee|_{J_Q}$ (in a multiplicative notation).

As was remarked above, if the rank of Φ is positive, a finite type Dynkin diagram of type J_Q fits uniquely as an excellent subdiagram J of the spectral diagram $\Sigma_s(\mathcal{R}, m)$ associated to G (this can be checked case-by-case), up to the action of $X^*_{\mathrm{un}}(G)$. Now we also assign weights to the nodes of $\mathtt{K} = I \backslash J$ as follows. By modifying ϕ_T (within its equivalence class) by an element of the Weyl group $W' = W(R'_0)$ we may assume that via $D\phi_T$ the affine simple reflections of $(R'_m)^{(1)}$ (relative to the base $(\mathcal{F}')^m$ of $(\mathcal{R}')^m$) correspond bijectively to the elements of $\mathtt{K}$.

This allows us to use $\mathtt{k} \in \mathtt{K}$ also to parameterize the elements of the base of $(R'_m)^{(1)}$. Let $\mathtt{k}_0 \in \mathtt{K}$ be the vertex of the unique (dual) affine simple root which is not in F'_m. From (T2), Proposition [54, Proposition 5.6](4) and Corollary [54, Corollary 5.8] (applied to the case $Q' = \emptyset$) we see that this is the unique element $\mathtt{k}_0 \in \mathtt{K}$ for which the corresponding vertex $D^a\phi_T(0) = \omega_{\mathtt{k}_0} \in C^\vee$ has the shortest length. We interpret the gradient $Da^\vee$ of a (dual) affine root $a^\vee \in R_m^{(1)}$ as a character on T (and similarly for dual affine roots of $(R'_m)^{(1)}$ on T'). By construction, the character lattice of L_n is mapped injectively to a sublattice of X_m^Q and injectively to a sublattice of X'_m. From (T3), [54, equation (8)] and considering the numerator of the μ-function (see Definition ([54, Definition 3.2]) it is easy to see that $D\phi_T^*(Da_{\mathtt{k}}^\vee)$ must be a rational multiple $D\phi_T^*(Da_{\mathtt{k}}^\vee) = f_{\mathtt{k}} Db_{\mathtt{k}}^\vee$ of a root $b_{\mathtt{k}}^\vee \in (F'_m)^{(1)}$. This sets up a bijection between the set of simple affine roots of $(F'_m)^{(1)}$ and the set $\mathrm{K} = I\backslash J$, and using this we will parameterize the elements of $(F'_m)^{(1)}$ also by the set K. By Proposition [54, Proposition 5.6](3) this bijection defines an isomorphism of affine reflection groups. By Proposition [54, Proposition 5.6](4) it is then clear that $b^\vee_{\mathrm{k}_0}$ has to be the extending affine root of the spectral diagram of $\mathcal{H}(R'_m, m')$.

And we can say more precisely, by considering the formula of the μ function of the Hecke algebra and (T3), that $f_{\mathrm{k}}^{-1} \in \mathbb{N}$, and that we can thus interpret the fraction $Db^\vee_{\mathrm{k}}$ as the character $f_{\mathrm{k}}^{-1} D\phi^*_T(Da^\vee_{\mathrm{k}})$ of T^Q. Now L itself is a coset of T^Q with origin $r_Q = s_Q c_Q$, and using the above remarks, it follows that for all $\mathrm{k} \in \mathrm{K}$, $Da^\vee_{\mathrm{k}}$ lifts to a constant multiple of a character of a suitable covering of T' (namely the fibered product of L and T' over L_n). We call this lift of $Da^\vee_{\mathrm{k}}$, expressed as a radical of $b^\vee_{\mathrm{k}}$, the *weight* w_{k} of $\mathrm{k} \in \mathrm{K}$. In view of (T3) and [54, equation (8)], and using Proposition [54, Proposition 5.6] we see that: $w_{\mathrm{k}} := \zeta_{\mathrm{k}} v^{c_{\mathrm{k}}} (Db^\vee_{\mathrm{k}})^{f_{\mathrm{k}}}$, where $\zeta_{\mathrm{k}} = 1$ if $\mathrm{k} \neq \mathrm{k}_0$, ζ_{k_0} is a $f_{\mathrm{k}_0}^{-1}$-th primitive root of 1, and $c_{\mathrm{k}} \in \mathbb{Z}$ (which can be computed by evaluating $Da^\vee_{\mathrm{k}}$ on c_Q). All this gives rise to the following notion:

Definition 3.3 Given a unipotent STM $\Phi : \mathcal{H}^{u,\tilde{\mathfrak{s}},e} \rightsquigarrow \mathcal{H}^{IM}$, the spectral diagram $\Sigma_s(\mathcal{R}, m)$ of G with the vertices of the excellent subdiagram J marked with the constant weights w_j, and the remaining vertices of K labelled with their weights w_{k} as above, is called the transfer map diagram of Φ.

Observe that $\prod_{i\in I} w_i^{n_i} = 1$, where $1 = \sum_{i\in I} n_i a_i^\vee$ is the decomposition of the constant function 1 as a linear combination of (dual) affine simple roots of $\mathcal{R}^m$ in terms of the base of simple roots $F_m^{(1)}$. In particular, there exists a constant C such that for all $\mathrm{k} \in \mathrm{K}$, $n_{\mathrm{k}} f_{\mathrm{k}} = C\tilde{n}_{\mathrm{k}}$, where $\sum_{\mathrm{k}\in\mathrm{K}} \tilde{n}_{\mathrm{k}} b^\vee_{\mathrm{k}} = 1$ is the decompostion of the constant function 1 in terms of the (dual) affine simple roots $(F'_m)^{(1)} = (\mathcal{F}')^m$ of $(\mathcal{R}')^m$. Clearly the value of $Da^\vee_{\mathrm{k}_0}$ on $s_Q = \omega_{\mathrm{k}_0}$ is a primitive n_{k_0}-th root of 1. Therefore we see that $C = 1$, and $f_{\mathrm{k}}^{-1} = n_{\mathrm{k}}/\tilde{n}_{\mathrm{k}}$ (this integer is called z_{k} by Lusztig [40, Section 2]); by Proposition [54, Proposition 5.6], we are in the setting of [40, Section 2] and we may therefore use the results of [40, Paragraph 2.11 to 2.14]. For example, by carefully analyzing the Cartan matrices it follows that if $b^\vee_k$, $b^\vee_{k'}$ are connected by a single edge, then $f_{\mathrm{k}}^{-1} = f_{\mathrm{k}'}^{-1}$. Moreover, f_{k}^{-1} is a divisor of $f_{\mathrm{k}_0}^{-1}$ for all $\mathrm{k} \in \mathrm{K}$, except possibly if $k, k_0 \in \mathrm{K}^\flat$, when we may have $f_{\mathrm{k}} f_{\mathrm{k}_0}^{-1} \in \{1, (1/2)^{\pm 1}\}$. We also note that, from the tables in [40, Section 7] and [44, Section 11], for all $\mathrm{k} \in \mathrm{K}$: $n_{\mathrm{k}}/n_{\mathrm{k}_0} \in \mathbb{Z}$. By the above it is clear that Φ is completely determined by its transfer map diagram.

The finite abelian group $K_L^n \subset T_L^{W_L}$ can be recovered from the transfer map diagram as the product over all $\mathrm{k} \in \mathrm{K}\backslash\{\mathrm{k}_0\}$ of cyclic groups C_{k} of order $z_{\mathrm{k}} = f_{\mathrm{k}}^{-1}$ (for those $\mathrm{k} \in \mathrm{K}\backslash\{\mathrm{k}_0\}$ for which $n_{m'}(b_{\mathrm{k}}) = 1$) or of order $z_{\mathrm{k}}/2$ (for $\mathrm{k} \in \mathrm{K}\backslash\{\mathrm{k}_0\}$ such that $n_{m'}(b_{\mathrm{k}}) = 2$ and z_{k} is even). For classical groups K_L^n is always trivial.

3.2 Main theorem

We finally have everything in place to formulate the two main theorem of this paper. Let $\mathbf{G}$ be a connected absolutely quasisimple K-split, k-quasisplit linear algebraic group. *For simplicity we will assume that G is of adjoint type.* Recall that $\mathcal{H}^{IM}$ denotes the Iwahori–Matsumoto Hecke algebra of $G = \mathbf{G}(K)$, i.e., the generic affine Hecke algebra $\mathcal{H}^{IM} = \mathcal{H}^{IM}(G)$ such that $\mathcal{H}^{IM}_{\mathbf{v}}$ is the Iwahori–Matsumoto Hecke algebra of $G(k) = G^F$ with respect to the standard cuspidal unipotent pair $\mathfrak{s}_0 := (\mathbb{B}, 1)$ where $\mathbb{B}$ denotes the Iwahori subgroup. Since $\mathfrak{s}_0$ is fixed for the action of $N\mathbb{B}^F/\mathbb{B}^F = \Omega_C^\theta$, the orbit $\mathcal{O}_0$ of $\mathfrak{s}_0$ equals $\mathcal{O}_0 = \{\mathfrak{s}_0\}$. We have $\Omega^{\mathfrak{s}_0,\theta} = \Omega_2^{\mathfrak{s}_0,\theta}$. Its trace τ^{IM} is normalized as in (25), i.e.,

$$\tau^{IM}(e^{IM}) = \det_V(\mathbf{v}\mathrm{Id}_V - \mathbf{v}^{-1}w_u\theta)^{-1} \tag{28}$$

where $V = \mathbb{R}Y$, θ denotes the action on Y of the outer automorphism of $G^\vee$ corresponding to F, and $w_u \in W_0$ is the image of $u \in \Omega_C \subset W$ under the canonical projection $W \to W_0$. Observe that $(\mathcal{H}^{IM}, \tau^{IM})$ is a direct summand of $\mathcal{H}_{\mathrm{uni}}(G)$, namely the unique summand of maximal rank. It corresponds to the Borel component of G^F, the Bernstein component corresponding to the cuspidal unipotent representation 1 of a minimal F-Levi subgroup M of G.

From Theorem 2.8 the group $X^*_{\mathrm{un}}(G)$ acts by (spectral) transfer automorphisms on $(\mathcal{H}_{\mathrm{uni}}(G), \tau)$. In particular $X^*_{\mathrm{un}}(G)$ acts by spectral automorphisms on $\mathcal{H}^{IM}$ too, (see Proposition [54, Proposition 3.5]) since $\mathcal{H}^{IM}$ is the unique summand of $\mathcal{H}_{\mathrm{uni}}$ of maximal rank.

3.2.1 Notational conventions for Hecke algebras Recall Definition [54, Definition 2.11] and recall that the spectral diagram can be expressed completely in terms of R_0 and of the W_0-invariant functions $m_\pm(\alpha)$ on R_0 defined by [54, Equation (4)]. In the proof of the theorem below, we will denote the unipotent normalized affine Hecke algebra of irreducible type $\mathcal{H}(\mathcal{R}^m, m)$, with X_m the weight lattice of the irreducible reduced root system R_m, as follows. If R_m is simply laced and the parameters $m_+(\alpha_{\mathrm{k}})$ are equal to $\mathfrak{b}$, we denote this unipotent Hecke algebra by $R_m[q^{\mathfrak{b}}]$. If R_m is not simply laced and not of type C_d, then we will denote this algebra by $R_m(m_+(\alpha_1), m_+(\alpha_2))[q^{\mathfrak{b}}]$ where $\alpha_1 \in F_m$ is long and α_2 is short, and $q^{\mathfrak{b}}$ is the base for the Hecke parameters (equivalently, we could write $R_m(\mathfrak{b}m_+(\alpha_1), \mathfrak{b}m_+(\alpha_2))[q]$). If both parameters are equal to $\mathfrak{b}$, we may also simply write $R_m[q^{\mathfrak{b}}]$, this will not create confusion. For $R_m = \mathrm{C}_d$, we will write $\mathrm{C}_d(m_-, m_+)[q^{\mathfrak{b}}]$ to denote the unipotent normalized affine Hecke algebra with $R_m = \mathrm{C}_d$ with $m_+(\alpha) = \mathfrak{b}$ for α a type D_d root of R_m, and for a short root β of B_d, we have $m_-(\beta) = \mathfrak{b}m_-$ and $m_+(\beta) = \mathfrak{b}m_+$. If $m_-(\beta) = 0$ and $m_+(\beta) = m_+(\alpha) = \mathfrak{b}$, then we may also denote this case $\mathrm{C}_d[q^{\mathfrak{b}}]$.

Theorem 3.4 *Let G be a connected, absolutely simple, quasisplit linear algebraic group of adjoint type, defined and unramified over a non-archimedean local field k. Let $\mathfrak{C}_{uni}(G)$ be the full subcategory of the spectral transfer category $\mathfrak{C}_{es}$ (with essentially strict STMs as morphisms) whose set of objects is the set of normalized unipotent affine Hecke algebras $\mathcal{H}^{u,\mathfrak{s},e}$ associated with the various inner forms G^u of G (where $u \in Z^1(F, G)$ runs over a complete set of representatives of the classes $[u] \in H^1(F, G)$). Let $\mathcal{H}_{uni}$ denote the direct sum of all the objects of $\mathfrak{C}_{uni}(G)$. Recall that there is a natural action of $X^*_{un}(G)$ on $\mathcal{H}_{uni}$ such that direct summands are mapped to direct summands, preserving the rank. In particular $X^*_{un}(G)$ acts on the unique summand $\mathcal{H}^{IM}(G)$ of $\mathcal{H}_{uni}$ of largest rank.*

There exists a $X^*_{un}(G)$*-equivariant STM*

$$\Phi : (\mathcal{H}_{uni}(G), \tau) \rightsquigarrow (\mathcal{H}^{IM}(G), \tau^{IM}) \tag{29}$$

which is essentially unique in the sense that if Φ' *is another such equivariant STM, then there exits a spectral transfer automorphism* σ *of* $(\mathcal{H}_{uni}(G), \tau)$ *such that* $\Phi' = \Phi \circ \sigma$.

The transfer map diagrams corresponding to the restrictions of Φ *to the various direct summands* $\mathcal{H}^{u,\tilde{\mathfrak{s}},e}$ *of* $\mathcal{H}_{uni}(G)$ *are equal to the corresponding geometric diagrams of* [40].

Corollary 3.5 *Recall that the spectral isogeny class of an object of* $\mathfrak{C}_{uni}(G)$ *is equal to its isomorphism class* [54, Proposition 8.3]*, and that these classes admit a canonical partial ordering* $\lesssim$ *as defined in* [54, Definition 8.2]. *Then* $(\mathcal{H}^{IM}(G), \tau^{IM}) \lesssim (\mathcal{H}, \tau)$ *for any object* $(\mathcal{H}, \tau)$ *of* $\mathfrak{C}_{uni}(G)$.

Theorem 3.4 is a consequence of the combined results of the following subsections.

3.2.2 The case of $G = PGL_{n+1}$ In this case, the only cuspidal unipotent representation comes from the anisotropic inner form $G^u = \mathbb{D}^\times/k^\times$ (where $\mathbb{D}$ is an unramified division algebra over k of rank $(n+1)^2$) and has a formal degree with q-rational factor given by fdeg $:= [n+1]_q^{-1}$ (cf. 2.2.3). It is obvious that there exists a unique cuspidal STM $(\mathbf{L}, \text{fdeg}) := A_0[q^{n+1}] \rightsquigarrow A_n[q]$, since $\mathrm{A}_n[q]$ has only one orbit of residual points (up to the action of $X^*_{\mathrm{un}}(G)$) and this has indeed the desired residue degree.

Based on this it is easy to construct the general STM for the unipotent types for this G, and prove that these are unique. Suppose we have a factorization $n+1 = (d+1)(m+1)$. Consider an inner form G^u of G such that u has order $m+1$. A maximal k-split torus $S \approx (k^\times)^d$ of G^u defines a Levi group $M^u = C_{G^u}(S)$ such that $M^u_{ssa} = M^u/S$ is of type $(\mathbb{D}^\times/k^\times)^{d+1}$ where $\mathbb{D}$ is an unramified division algebra over k of rank $(m+1)^2$. Then J is of type A_m^{d+1}, which fits in a unique sense (up to diagram automorphisms as usual) as an essential subdiagram of the spectral diagram of $\mathcal{H}^{IM}$. The Hecke algebra $\mathcal{H}^{u,\mathfrak{s},u}$ is of type $\mathrm{A}_d[q^{m+1}]$. For the unique *strict* STM we make sure that J does not contain $a_0^\vee$. The weights for the vertices of J are equal to q, and for those of $\mathtt{k} \in I \setminus J$ equal to $q^{-m}Db_{\mathtt{k}}$. It is an easy check that this indeed defines a strict STM $\mathrm{A}_d[q^{m+1}] \rightsquigarrow \mathrm{A}_n[q]$.

The uniqueness of such strict STM up to $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{H}^{u,\mathfrak{s},e})$ is clear as before: Any strict STM $\phi : \mathrm{A}_d[q^{m+1}] \rightsquigarrow \mathrm{A}_n[q]$ is obtained by induction of a cuspidal one for M_{ssa}, which determines J and the underlying geometric diagram of ϕ as before. There assignment of weights to the vertices of $\mathtt{K}$ is dictated by the basic properties of an STM as explained above. Hence ϕ must be equal to the STM constructed above, up to a diagram automorphism of A_d. By Theorem 2.8 the direct summands of $\mathcal{H}^{u,\mathfrak{s}}$ form a torsor for $(\Omega_1^{\mathfrak{s},e})^* = \Omega^*/(\Omega_2^{\mathfrak{s},e})^* \approx C_{m+1}$, and hence there is a unique way to write down a Ω^*-*equivariant* STM $\mathcal{H}^{u,\mathfrak{s}} \rightsquigarrow \mathcal{H}^{IM}$, up to the action of $\mathrm{Aut}_{\mathrm{es}}(\mathcal{H}^{u,\mathfrak{s}})$. This completely finishes the proof for the case of PGL_{n+1}.

3.2.3 Existence and uniqueness of rank 0 *STMs for exceptional groups* This is a case check (with some help of Maple, to simplify the product formulas for the q-factor of the formal degree as given in [56]), almost all of which has already been done in the existing literature. Let G be an k-quasisplit adjoint group over with k which is split over K, of type ${}^3\mathrm{D}_4$, E_6, ${}^2\mathrm{E}_6$, E_7, E_8, F_4, G_2. One uses the classification of the residual points and the product formula for the q-rational factor of the formal degree from [56] to compute, for each orbit $W_0 r$ of generic residual points (in the sense of the present paper), the q-rational factor of the residue degree $\mu^{IM,(r)}(r)$ for the Iwahori–Matsumoto Hecke algebra $\mathcal{H}^{IM}$ of G. (Many of these results are already in the literature; For E_8 this list was given in [21] using essentially the same method. For all split exceptional groups this list can be found in [60].

Note that the computations in [60] can be simplified a lot using the classifications and the *product formula* from [56], to just "clearing q-fractions", since our formal degree formula is already given in "product form" (as opposed to an alternating sum of rational functions as in [60]). Also note that we are for this list only interested in the Iwahori spherical case.) We note that these lists reveal that these residues of μ^{IM} at distinct orbits $W_0 r \neq W_0 r'$ are distinct for all exceptional cases. Hence in the exceptional cases the uniqueness (up to diagram automorphisms) of rank 0 spectral transfer maps for irreducible unipotent Hecke algebras is guaranteed by this.

The existence of the desired cuspidal unipotent STMs is now an easy task; one considers the list of all cuspidal unipotent representations of all inner forms of G. This means that we need to make a list of all maximal F_u-stable parahoric subgroups of the inner forms G^u, consider their reductive quotients over $\mathbf{k}$, and for those quotients which admit a cuspidal unipotent character, compute the normalization of the associated Hecke algebra $\mathcal{H}^{u,\tilde{\mathfrak{s}},e} := (\mathbf{L}, \tau^{\mathfrak{s},e})$ according to (25). Of course the main part of this formula is the degree of the unipotent cuspidal characters of the simple finite groups of Lie type, which is due to Lusztig [32–34] and conveniently tabulated in [9]. Finally we need to see if the q-rational parts of these expressions show up in our list of residues of the μ-function. This indeed leads to cuspidal transfer map diagrams with the same underlying sets J as listed by Lusztig in [40] and [43], and for each of those diagrams, there exists one generic linear residual point for J (in the form of the collection of weights assigned to the vertices of J) producing the correct residue of μ and thus an STM.

Let us give the results for the two non-split quasisplit cases which were not yet treated in the existing literature. The unipotent Hecke algebra $\mathrm{G}_2(3,1)[q]$ (for ${}^3\mathrm{D}_4$), and $\mathrm{F}_4(2,1)[q]$ (for ${}^2\mathrm{E}_6$). The first case $\mathrm{G}_2(3,1)[q]$ has 4 residual points. The spectral diagram [54] of this Hecke algebra is the untwisted version of the Kac diagram [61, Subsection 4.4], with the equal parameters $3k$ attached to the nodes (a similar remark applies to all simple quasisplit unramified cases).

Let us use the maximal subdiagrams of this Kac diagram to name the various orbits of residual points. There are two orbits of residual points G_2 and $\mathrm{G}_2(a_1)$ with positive central character. The corresponding groups A_λ [where λ denotes the corresponding discrete unramified Langlands parameters via equation (17)] are 1 and S_3 respectively. There are two nonreal orbits A_2 (with $A_\lambda = 1$) and $\mathrm{A}_1 \times \mathrm{A}_1$ (with $A_\lambda = C_2$). Looking at the q-rational factor of the residue of the μ function at these points, we find that the cuspidal (orbits of) residual points are $\mathrm{G}_2(a_1)$ (matching the degree of ${}^3\mathrm{D}_4[1]$) and $\mathrm{A}_1 \times \mathrm{A}_1$ (matching the degree of ${}^3\mathrm{D}_4[-1]$). Together with the Iwahori spherical unipotent discrete series these cases make up for the set of 7 unipotent discrete series (5 of which are Iwahori spherical, while the others are cuspidal).

A similar discussion for $\mathrm{F}_4(2, 1)[q]$ shows the following. Again we use the Kac diagram [61, Subsection 4.5], this time with the constant parameters $2k$ attached to each node to indicate the various orbits of residual points. We have 9 orbits of residual points (in the notation of [56]). There are 4 orbits with positive real central character, corresponding to $\Psi(D_i(2k, k)) = D_i(2k, 2k)$ with $D_i(x, x)$ (for $i = 1, \dots, 4$) as listed in [56, Table 3] (or equivalently, the $D_i(x, x)$ are the weighted Dynkin diagrams $\mathrm{F}_4, \mathrm{F}_4(\sigma_1), \mathrm{F}_4(\sigma_2), \mathrm{F}_4(\sigma_3)$ (in this order) of the distinguished nilpotent orbits of F_4 as denoted by [9])). The corresponding groups A_λ are of the form $1,\ S_2,\ S_2,\ S_3$ respectively. In addition there is 1 orbit corresponding to $\mathrm{A}_1 \times \mathrm{B}_3$ with $A_\lambda = C_2$, 1 orbit for $\mathrm{A}_2 \times \mathrm{A}_2$ with $A_\lambda = C_3$, 1 orbit for $\mathrm{A}_3 \times \mathrm{A}_1$ with $A_\lambda = C_2$, and finally 2 orbits corresponding to C_4 with $A_\lambda = 1$ (the regular orbit) and $A_\lambda = C_2$ (the subregular orbit) respectively. The cuspidal orbits of residual points are in this case the ones corresponding to $\mathrm{F}_4(\sigma_3)$ (matching the degree of ${}^2\mathrm{E}_6[1]$) and the one of type $\mathrm{A}_2 \times \mathrm{A}_2$ (matching the degree of ${}^2\mathrm{E}_6[\theta]$ and of ${}^2\mathrm{E}_6[\theta^2]$). Hence we expect in total 18 unipotent discrete series in this case (corresponding to the irreducible representations of the various A_λ). Using the classification of [56, Theorem 8.7] we can identify 13 Iwahori spherical cases (corresponding to the discrete spectrum of $\mathrm{F}_4(2, 1)[q]$), and there are 3 cuspidal ones. (The two missing ones are of intermediate type, corresponding to a rank 1 STM. See paragraph 3.2.5.) This agrees with the tables in [40] and [43].

3.2.4 Existence of STMs for the exceptional cases Let us now consider the existence of the positive rank STMs in the exceptional cases. Let S^u be a k-spit torus. As always, we assume that $S^u \subset S$, with $S \subset G$ a fixed maximal k-split torus. Consider $M = C_G(S^u)^0$, $M^u = C_G(S^u)^0$, and assume that $M^u_{ssa} = M^u/S^u$ admits a cuspidal unipotent character σ^u. Recall that $\mathcal{H}^{IM}(M_{ssa}) \approx \mathcal{H}^{IM}_Q$ for some proper subset $Q \subset F_m$. In particular, at most one of the irreducible components of Q will not be of type A, and possible irreducible factors of type A have to be in the anisotropic kernel of G^u. By the results for type A and for rank 0 STMs for irreducible exceptional types, there exists a unique rank 0 STM $\psi : \mathbf{L} \rightsquigarrow \mathcal{H}^{IM}(M_{ssa})$ for the cuspidal unipotent representation σ^u. Our task will be to see that this STM map can be induced to $\mathcal{H}^{IM}$.

As in 3.1.2, consider $L = r_M T^J$ where $r_M = s_M c_M \in T_M$ is the image of ϕ, $J \subset \Sigma_s(\mathcal{R}, m)$ is the excellent subset of type J_M associated to the STM diagram of ϕ. Here c_M is in dominant position with respect to J, so that the weight of a root $a_i^\vee$ in J is given $Da_i^\vee(c_M)$, and s_M is a vertex of $C^\vee$ in $F_m^{(1)} \backslash J$. By what was said in the previous paragraph it follows that these diagrams are exactly the exceptional geometric diagrams of Lusztig, with weights attached to the vertices of the boxed set of vertices J. We remark that for all exceptional cases, the geometric diagrams with J such that $|\mathtt{K}| > 1$ (i.e., of positive rank), the components of J are all of type A (this simplicity is in remarkable contrast with the classical cases). Therefore the weights w_j with $j \in J$ are simply equal to $q^{m_R^\vee(a_j^\vee)}$. If there would indeed exist a corresponding transfer map, then its transfer map diagram should be obtained by assigning in addition weights to the vertices in $\mathtt{K} = F_m^{(1)} \backslash J$, as described in Definition 3.3. These weights turn out to be uniquely determined by the basic property Proposition [54, Proposition 5.2] of spectral transfer maps (applied to the case of residual points), and this also enables us to find these weights w_i easily (using the known classification of residual points of [22] and [56]). Our task is then to prove that these eligible diagrams thus obtained are indeed transfer map diagrams.

In order to do so, we need to first find $\mathtt{k}_0$. This has to be the unique vertex $\mathtt{k} \in \mathtt{K}$ of the geometric diagram such that the corresponding vertex $\omega_{\mathtt{k}} \in C^\vee$ has the shortest length. It is easy to check in all exceptional geometric diagrams that this condition defines a unique vertex $\mathtt{k}_0 \in \mathtt{K}$. The cuspidal unipotent representation σ_u of M_{ssa} lifts to a cuspidal unipotent representation $\tilde{\sigma}_u$ of M, and the cuspidal pair $(M, \tilde{\sigma}_u)$ is obtained by compact induction from a cuspidal unipotent type $\mathfrak{s} := (\mathbb{P}_J, \delta)$. The affine Hecke algebra $\mathcal{H}^{u,\mathfrak{s},e}$ of the cuspidal unipotent type is given, and let $\mathcal{R}'_{m'}$ be the corresponding based root datum with multiplicity function m'.

Next we need to determine the bijection between the affine simple roots $b_i^\vee$ of the spectral diagram of $\mathcal{H}^{u,\mathfrak{s},e}$ and of $\mathtt{K}$. This was done by Lusztig: According to the main result of [40], there exists a matching such that $\mathtt{k}_0$ corresponds to b_0, and such that the underlying affine Coxeter diagram of the spectral diagram of $\mathcal{H}^{u,\mathfrak{s},e}$ matches the Coxeter relations of the reflections in the quotient roots $\overline{\alpha}_{\mathtt{k}} = Da_{\mathtt{k}}^\vee|L$ (cf. [40, 2.11(c)]). (Here we identify L with $T^J \subset T$, the maximal subtorus on which the gradients of the roots from J are constant, by choosing $r_M \in L$ as its origin (given by the weights of $j \in J$ and s_0 corresponding to $\omega_{\mathtt{k}_0}$)). Since this matching is only based on the underlying affine Weyl groups, and by Proposition [54, Proposition 5.6], it is clear that a possible spectral map diagram has to provide the same matching. It is easy to check case-by-case that such a matching is unique up to diagram automorphisms preserving the parameters $m_{R'}^\vee$ of the spectral diagram of $\mathcal{H}^{u,\mathfrak{s},e}$. Thus we fix such a matching, and use this to also parameterize the (dual) affine simple roots of the spectral diagram of $\mathcal{H}^{u,\mathfrak{s},e}$ by $\mathtt{k} \in \mathtt{K}$.

Following notations as in 3.1.4, we need to assign an integer $c_{\mathtt{k}}$ to each node $\mathtt{k} \in \mathtt{K}$, in order to define the weights $w_{\mathtt{k}}$ for all $\mathtt{k} \in \mathtt{K}$. We define $c_{\mathtt{k}}$ by the formula

$$c_{\mathtt{k}} = m_R^\vee(a_{\mathtt{k}}^\vee) - f_{\mathtt{k}} m_{R'}^\vee(b_{\mathtt{k}}^\vee) \tag{30}$$

where $a_{\mathtt{k}}^\vee$ denotes the (dual) affine root of $\mathcal{R}^m$ associated with $\mathtt{k}$, and $b_{\mathtt{k}}^\vee$ the corresponding (dual) affine root of the spectral diagram of $\mathcal{H}^{u,\mathfrak{s},e}$.

The diagram thus obtained defines a map ϕ from T' to a suitable quotient of $L \subset T$. For each maximal proper subdiagram D of a spectral diagram of a semi-standard affine Hecke algebra $\mathcal{H}(\mathcal{R}, m)$ there exists a generic residual point r_R^D such that $Da_{\mathtt{k}}^\vee(r_R^D) = v^{2m_R^\vee(a_{\mathtt{k}})}$ for all $\mathtt{k} \in D$, and for $\mathtt{k} \in \mathtt{K}\backslash D$, such that $b_{\mathtt{k}}^\vee(r_R^D|v = 1)$ is a primitive root of 1 of order $n_{\mathtt{k}}$. The above assignment means that we require the alleged spectral transfer map ϕ to satisfy the property that $\phi(r_{R'}^D) = r_R^{D\cup J}$. We can check easily case-by-case that this map then also sends all other residual points of $\mathcal{H}^{u,\mathfrak{s},e}$ to residual points of $\mathcal{H}^{IM}$, and that these weights are the only possible weights defining a map with such properties.

Remark 3.6 Thus, the image under ϕ_Z of the central character of the one-dimensional discrete series representation of $\mathcal{H}^{u,\mathfrak{s},e}$ which is the deformation of the sign character of its underlying affine Weyl group is equal to the central character of $\mathcal{H}^{IM}(G)$ of the analogous one dimensional character. If $\mathcal{H}^{u,\mathfrak{s},e}$ is the Iwahori–Hecke algebra of an inner twist G^u of G, then this is true in general, since we know that the formal degree of the Steinberg character is unchanged by inner twists. For exceptional groups it is true for all unipotent STMs of positive rank, which seems related to the fact that for these STMs, the subset $J \subset F_m^{(1)}$ consists of type A components only. In classical cases STMs do not have this property in general.

Finally we need to check that the map ϕ we have thus defined indeed defines an STM. This amounts to applying ϕ^* to $(\mu^{IM})^{(L)}$, making the substitutions $\phi^*(\alpha_i) = w_i$ for all $i \in I\backslash\{0\}$, and checking that this equals the μ-function $\mu^{(u,\mathfrak{s},e)}$ of $(\mathcal{H}^{u,\mathfrak{s},e}, d^{\tau,\mathfrak{s},e})$ up to a rational constant. Now this is already clear for the constant factor of $d^{\tau,\mathfrak{s},e}$ because of our choice of the weights of the $j \in J$ and the fact that we started out from a cuspidal STM for σ^u for M^u_{ssa}. Hence we only need to consider, for all $\mathtt{k} \in \mathtt{K}$, the cancellations in $\phi^*((\mu^{IM})^{(L)})$ for the factors in the numerator and denominator which are of the form $(1 - \zeta v^A (Db_{\mathtt{k}}^\vee)^F)$ (with A, F rational, F nonzero, and ζ a root of unity).

This is a tedious but simple task: We need to compile the table of all positive roots $\alpha \in R_{m,+}$, consider those α such that $\overline{\alpha} = \alpha|_L$ is a nonzero multiple of $\overline{\alpha}_{\mathtt{k}}$ (upon ignoring the coefficients of α at the $j \in J$, and using the relation $(\sum_{\mathtt{k}\in\mathtt{K}} n_{\mathtt{k}} n_{\mathtt{k}_0}^{-1} Da_{\mathtt{k}}^\vee)|_L = \zeta_{\mathtt{k}_0} v^l$ (with $\zeta_{\mathtt{k}_0}$ a primitive root of 1 of order $n_{\mathtt{k}_0}$ and $l \in \mathbb{Z}$) which follows from $\sum_{\mathtt{k}\in\mathtt{K}} \tilde{n}_{\mathtt{k}} Db_{\mathtt{k}}^\vee = 1$ and the discussion in paragraph 3.1.4). Then we compute for each of those roots the value $\phi^*(\alpha)$. This produces a list of integral multiples of $f_{\mathtt{k}} Db_{\mathtt{k}}^\vee$, and for each member of that list, a list of values of the form $\zeta_j v^i$ with ζ_j a root of 1 (of order divisible by $z_{\mathtt{k}_0}$), and v^i an integral power of v. From these lists we can easily see the cancellations of these type of factors in $\phi^*(\mu^{IM})^{(L)})$, and check that a rational function of the form

$$\frac{(1-\beta^2)^2}{(1+v^{-2m_-(\beta)}\beta)(1+v^{2m_-(\beta)}\beta)(1-v^{-2m_+(\beta)}\beta)(1-v^{2m_+(\beta)}\beta)} \tag{31}$$

(with $\beta = Db_{\mathtt{k}}^\vee$) remains, as desired. In this way we verify that all the diagrams so obtained are spectral map diagrams of spectral transfer maps, in all cases.

As a (rather complicated) example, let us look at $\tilde{E}_8/\mathrm{A}_3\mathrm{A}_3\mathrm{A}_1$. This diagram arises by induction from the cuspidal pair (E_7, σ^u), whose spectral map diagram is given by $\tilde{\mathrm{E}}_7/\mathrm{A}_3\mathrm{A}_3\mathrm{A}_1$ (see the geometric diagram of [40, 7.14]). The spectral diagram of $\mathcal{H}^{u,\mathfrak{s},e}$ is of type $\mathrm{C}_1(7/2, 4)[q]$. The vertex $\mathtt{k}_0$ is labelled by 1 in [40, 7.8]. We write the simple roots of $\mathrm{C}_1(7/2, 4)[q]$ in the form $b_1^\vee = 1-2\beta$, and $b_2^\vee = 2\beta$. The weights of the roots $a_1 := \alpha_6$ and $a_2^\vee := \alpha_3$ are $w_1 = \sqrt{-1}v^{-6}(-\beta/2)$ and $w_2 = v^{-7}\beta/2$. When α runs over the positive roots of E_8 such that $\phi^*(\alpha)$ is a nonzero multiple of $\beta/2$, the following lists of factors in front of $\beta/2$ appear: For $\zeta := \pm\sqrt{-1}$, the following powers of v: $v^{\pm 6}$, 2 times $v^{\pm 4}$, 3 times $v^{\pm 2}$, and 4 times 1, and for $\zeta := \pm 1$, the following powers of v: $v^{\pm 5}$, 2 times $v^{\pm 3}$, 3 times $v^{\pm 1}$. In addition the restricted root β appears, with factor 1. One easily checks that this indeed produces the μ function of $\mathrm{C}_1(7/2, 4)[q]$. The group K_L^n is isomorphic to C_2 (caused by taking the square root of β). Note that this is equal to the central subgroup $T_L^{W_L}$ with $T_L \subset T$ the subtorus whose cocharacter lattice is the coroot lattice of $\mathrm{E}_7 \subset \mathrm{E}_8$.

As an example of a somewhat different kind, let us look at the unramified nontrivial inner form ${}^2\tilde{\mathrm{E}}_7$ of type E_7 (cf. [40, 7.18]). This case is induced from the trivial representation σ^u of the anisotropic kernel M^u_{ssa} of the group of type ${}^2\tilde{\mathrm{E}}_7$, which is an anisotropic reductive group of rank 3. The spectral diagram of $\mathcal{H}^{u,\emptyset,e}$ is of type $\mathrm{F}_4(1,2)[q]$. Since F_u has order 2, it follows that [see 2.2.3, 2.2.2, and (24)] the q-rational factor of the formal degree of σ^u is $[2]_q^{-3}$. This corresponds to the residue degree of the μ-function of a Hecke algebra of type $\mathrm{A}_1[q]^3$ at its unique residue point. Hence we need to take J of type $\mathrm{A}_1\mathrm{A}_1\mathrm{A}_1$. This subdiagram fits in a unique way as an excellent subset in the spectral diagram of type $\tilde{E}_7$, up to the diagram autmorphism of $\tilde{E}_7$. However, we need to choose the unique such embedding of J such that the root $a_0^\vee$ does not belong to J (i.e., $J \subset F_{m,0}$ here; it is easy to check that the other possibility does not lead to a *strict* STM (although it does lead to an essentially strict but non-strict STM, obtained by composing the strict STM we are about to construct by the nontrivial diagram automorphism of the spectral diagram of type E_7, cf [54, Remark 6.2])). Since we know that a transfer map diagram which is induced from this cuspidal pair must have the property that J appears as an excellent subset of the diagram of, it is clear that Lusztig's geometric diagram for [40, 7.18] indeed should be the underlying geometric diagram of a spectral transfer map (if it exists), and k_0 is the vertex numbered by 5 in [40, 7.18]. The f_{k} are all equal to 1, and (in the numbering of [40, 7.18]) we have $w_i = q^{\lambda_i} Db_i^\vee$ with $\lambda_i = -1$ for $i = 1, 2$ and 0 for $i = 3, 4, 5$. It is easy to check that this gives a spectral transfer map Φ. All other examples are done similarly by executing this algorithm. We remark that $z_{\mathrm{k}} \leq 3$ in all cases, except possibly when $Db_{\mathrm{k}}^\vee$ is a divisible root of R'_m, when $z_{\mathrm{k}} = 4$ may occur (as in the above example). We leave it to the reader to check the remaining exceptional cases by him/herself.

3.2.5 The exceptional non-split quasisplit cases For convenience we explicitly list the unipotent STMs for the non-split quasisplit cases ${}^3\mathrm{D}_4$ and ${}^2\mathrm{E}_6$. Both groups do not have nontrivial inner forms. The rank 0 STMs were all described in paragraph 3.2.3. For the case ${}^3\mathrm{D}_4$, up to G^F-conjugacy, the only F-stable cuspidal unipotent pairs $(\mathbb{P}, \sigma)$ are those with $\mathbb{P}$, an F-stable Iwahori subgroup, and $\sigma = 1$, or with $\mathbb{P}$ maximal hyperspecial. Thus, the only nontrivial unipotent STMs are the rank 0 ones which were already described in paragraph 3.2.3

For ${}^2E_6$, besides the rank 0 cases already described in paragraph 3.2.3, we have the rank 1 STM which arises from the cuspidal unipotent pair $(\mathbb{P}, \sigma)$ where $\overline{\mathbb{P}}$ is of type ${}^2A_5$ (and σ its unique cuspidal unipotent representation). This gives rise to a unipotent affine Hecke algebra of type $\mathrm{C}_1(4,5)[q]$. The unique STM $\Phi : \mathrm{C}_1(4,5)[q] \rightsquigarrow \mathrm{F}_4(2,1)[q]$ maps the two central characters of the two discrete series of $\mathrm{C}_1(4,5)[q]$ in a unique way to two orbits of residual points of $\mathrm{F}_4(2,1)[q]$. Namely, q^5 maps to $\mathrm{A}_1 \times \mathrm{B}_3$, while $-q^4$ maps to $\mathrm{A}_3 \times \mathrm{A}_1$. More precisely, Φ can be represented by a morphism $\phi : T_1 \to L_n$ of torsors of the algebraic tori. Here we consider the algebraic tori T_i associated to the two relevant affine Hecke algebras (with T_i of rank i, and with coordinates given by the simple roots β_1 for T_1 and $\alpha_1, \ldots, \alpha_4$ for T_4, with α_3, α_4 the short simple roots). Further $L \subset T_4$ is a rank 1 residual coset given by the equations $(\alpha_1 + 2\alpha_3) = -q^{-5}$, $\alpha_2 = q^2$, $\alpha_4 = q$, while L_n is a quotient $L \to L_n$ of L, a double cover. The morphism ϕ can be chosen as follows. Let $F_{m'} = \{\beta_0, \beta_1\}$. We check that $\mathrm{k}_0 = 1$ and $\mathrm{k}_1 = 3$, and that the additional weights of the transfer diagram map of ϕ are given by $w_1 = -v^{-4}\beta_0^{1/2}$ and $w_3 = v^{-3}\beta_1^{1/4}$. Together with the information in paragraph 3.2.3, this completes the descriptions of the relevant STMs $\mathcal{H}^{u,\mathfrak{s}} \rightsquigarrow \mathcal{H}^{IM}$ for the cases ${}^3\mathrm{D}_4$ and ${}^2\mathrm{E}_6$.

Remark 3.7 In these two cases ${}^3\mathrm{D}_4$ and ${}^2\mathrm{E}_6$ we see that a parameterization of the unipotent discrete series representations is completely determined by the matching condition that the q-rational factor of the formal degree needs to equal the residue $\mu^{IM,(\{r\})}(r)$, together with the requirement that we assign the generic representation to the trivial representation of A_λ (where λ is the unramified Langlands parameter which corresponds to W_0r according to (17)). (To be precise, in the case ${}^2\mathrm{E}_6$ this fixes the parameterization except for the interchangeability of ${}^2\mathrm{E}_6[\theta]$ and ${}^2\mathrm{E}_6[\theta^2]$.)

3.2.6 Unipotent affine Hecke algebras of type $C_d(m_-, m_+)$ For an absolutely simple, quasisplit classical group G of adjoint type other than PGL_{n+1}, the proof of the essential uniqueness of an STM $\phi : \mathcal{H}^{u,\mathfrak{s},e} \rightsquigarrow \mathcal{H}^{IM}(G)$ for an affine Hecke $\mathcal{H}^{u,\mathfrak{s},e}$ of any unipotent type $\mathfrak{s}$ for any inner form G^u follows the same pattern as in the exceptional case, by reducing the statement to the essential uniqueness for cuspidal STMs. The proof of the existence of an STM ϕ as above is treated quite differently however, for most cases by generating ϕ as a composition of a small number of basic STMs which generate the spectral transfer category whose objects consist of *all* unipotent affine Hecke algebras of the form $\mathcal{H}^{u,\mathfrak{s},e}$ for all groups in certain classical families (containing G). It turns out that in essence there are only 2 types of basic building blocks generating almost all STMs between the unipotent affine Hecke algebras associated to the unitary, orthogonal and symplectic groups. Apart from the STMs built from these basic generators, there is one additional, very important type of basic STMs of the form $\phi : \mathcal{H}^{u,\mathfrak{s},e} \rightsquigarrow \mathcal{H}^{IM}(G)$ for the orthogonal and symplectic cases which we call *extraspecial*.

As mentioned above, we will now first define some basic building blocks of STMs between classical affine Hecke algebras which are associated to the unitary, orthogonal and symplectic groups. We define a category $\mathfrak{C}_{\mathrm{class}}$ whose objects are normalized affine Hecke algebras of type $(\mathrm{C}_d(m_-, m_+)[q^{\mathfrak{b}}], \tau_{m_-,m_+})$ where $d \in \mathbb{Z}_{\geq 0}$, $(m_-, m_+) \in V$, the set of ordered pairs (m_-, m_+) of elements $m_\pm \in \mathbb{Z}/4$ satisfying $m_+ - m_- \in \mathbb{Z}/2$, and $\mathfrak{b} = 1$ if both $m_+ - m_- \in \mathbb{Z}$ and $m_+ + m_- \in \mathbb{Z}$, otherwise we put $\mathfrak{b} = 2$. Hence the objects of $\mathcal{C}_{\mathrm{class}}$ are in bijection with the set V of triples $(d; (m_-, m_+))$ as described above.

The trace $\tau = \tau_{m_-,m_+}$ is normalized as follows. First, we decompose V in six disjoint subsets V^X with $X \in \{\mathrm{I, II, III, IV, V, VI}\}$, which are defined as follows. If $m_\pm \in \mathbf{Z} \pm \frac{1}{4}$, write $|m_\pm| = \kappa_\pm + \frac{(2\epsilon_\pm - 1)}{4}$ with $\epsilon_\pm \in \{0, 1\}$ and $\kappa_\pm \in \mathbb{Z}_{\geq 0}$. Define $\delta_\pm \in \{0, 1\}$ by $\kappa_\pm \in \delta_\pm + 2\mathbb{Z}$. Then we define:

$$\begin{aligned}
&(d; (m_-, m_+)) \in V^{\mathrm{I}} && \text{iff } m_\pm \in \mathbb{Z}/2 \text{ and } m_- - m_+ \notin \mathbb{Z},\\
&(d; (m_-, m_+)) \in V^{\mathrm{II}} && \text{iff } m_\pm \in \mathbb{Z} + \tfrac{1}{2} \text{ and } m_- - m_+ \in \mathbb{Z},\\
&(d; (m_-, m_+)) \in V^{\mathrm{III}} && \text{iff } m_\pm \in \mathbb{Z} \text{ and } m_- - m_+ \notin 2\mathbb{Z},
\end{aligned}$$

$$\begin{array}{lll}(d;(m_-,m_+))\in V^{\mathrm{IV}} & \text{iff } m_\pm\in\mathbb{Z} \text{ and } m_--m_+\in 2\mathbb{Z}, \\ (d;(m_-,m_+))\in V^{\mathrm{V}} & \text{iff } m_\pm\in\mathbb{Z}\pm\frac{1}{4} \text{ and } \delta_--\delta_+\neq 0, \\ (d;(m_-,m_+))\in V^{\mathrm{VI}} & \text{iff } m_\pm\in\mathbb{Z}\pm\frac{1}{4} \text{ and } \delta_--\delta_+=0.\end{array} \tag{32}$$

Observe that the type X of $(d;(m_-,m_+))$ only depends on (m_-,m_+); we will often simply write $(m_-,m_+)\in V^{\mathrm{X}}$ instead of $(d;(m_-,m_+))\in V^{\mathrm{X}}$. We now normalize the traces τ_{m_-,m_+} as follows. These traces are of the form $\tau_{m_-,m_+}=(v^{\mathfrak{b}}-v^{-\mathfrak{b}})^{-d}\tau^0_{m_-,m_+}$, where $\tau^0_{m_-,m_+}$ is independent of the rank d (and d is suppressed in the notation). Explicitly we define τ_{m_-,m_+} by

$$d^{\tau}_{m_-,m_+}=(v^{\mathfrak{b}}-v^{-\mathfrak{b}})^d\tau_{m_-,m_+}(1):=\begin{cases} d_a^{\tau,\{^2A\}}(q)d_b^{\tau,\{^2A\}}(q) & \text{if } (m_-,m_+)\in V^{\mathrm{I}} \\ d_a^{\tau,\mathrm{D}}(q)d_b^{\tau,\mathrm{B}}(q) & \text{if } (m_-,m_+)\in V^{\mathrm{II}} \\ d_a^{\tau,\mathrm{B}}(q)d_b^{\tau,\mathrm{B}}(q) & \text{if } (m_-,m_+)\in V^{\mathrm{III}} \\ d_a^{\tau,\mathrm{D}}(q)d_b^{\tau,\mathrm{D}}(q) & \text{if } (m_-,m_+)\in V^{\mathrm{IV}} \\ d_a^{\tau,\{^2A\}}(q)d_b^{\tau,\mathrm{B}}(q^2) & \text{if } (m_-,m_+)\in V^{\mathrm{V}} \\ d_a^{\tau,\{^2A\}}(q)d_b^{\tau,\mathrm{D}}(q^2) & \text{if } (m_-,m_+)\in V^{\mathrm{VI}}. \end{cases} \tag{33}$$

Here $d_s^{\tau,\{^2A\}}(q)$ is the q-rational part of the formal degree of a cuspidal unipotent character for the adjoint group G of type $^2\mathrm{A}_l$ which is compactly induced from the unique cuspidal unipotent character of a maximal parahoric subgroup whose reductive quotient is of type $^2\mathrm{A}_l(q^2)$, with $l=\frac{1}{2}(s^2+s)-1$ (with $s\in\mathbb{Z}_{\geq 1}$) (see Proposition 2.5; it is convenient to extend this to $s=0$ by setting $d_0^{\tau,\{^2A\}}=1$); similarly $d_s^{\tau,\mathrm{B}}(q)$ denotes the q-rational part of the formal degree of a cuspidal unipotent representation of G of type B_l induced in this sense from $\mathrm{B}_l(q)$ with $l=s^2+s$ (with $s\in\mathbb{Z}_{\geq 0}$) (this degree covers the cuspidal character of the odd orthogonal and the symplectic groups); $d_s^{\tau,\mathrm{D}}(q)$ denotes the q-rational part of the formal degree of a cuspidal unipotent representation of G of type D_l induced from $\mathrm{D}_l(q)$, with $l=s^2$ (with $s\in\mathbb{Z}_{\geq 0}$) (this degree covers the cuspidal character of the even split orthogonal groups (s even) and of the even quasisplit orthogonal groups (s odd)). (Using [9, Section 13.7] and (25) it is easy to give explicit formulas for these formal degrees.) where the set $\{a,b\}$ with $a,b\in\mathbb{Z}_{\geq 0}$ is determined by the following equalities of sets:

$$\begin{array}{ll}\{\frac{1}{2}+a,\frac{1}{2}+b\}=\{|m_+-m_-|,|m_++m_-|\} & \text{if } (m_-,m_+)\in V^{\mathrm{I}} \\ \{2a,1+2b\}=\{|m_+-m_-|,|m_++m_-|\} & \text{if } (m_-,m_+)\in V^{\mathrm{II}} \\ \{1+2a,1+2b\}=\{|m_+-m_-|,|m_++m_-|\} & \text{if } (m_-,m_+)\in V^{\mathrm{III}} \\ \{2a,2b\}=\{|m_+-m_-|,|m_++m_-|\} & \text{if } (m_-,m_+)\in V^{\mathrm{IV}} \\ \{\frac{1}{2}+a,1+2b\}=\{|m_+-m_-|,|m_++m_-|\} & \text{if } (m_-,m_+)\in V^{\mathrm{V}} \\ \{\frac{1}{2}+a,2b\}=\{|m_+-m_-|,|m_++m_-|\} & \text{if } (m_-,m_+)\in V^{\mathrm{VI}}.\end{array} \tag{34}$$

This determines a and b in case II, V, VI, and it determines a and b up-to-order in the other cases, so that the normalization (33) is always well defined.

Now we define the building blocks of the STMs between these affine Hecke algebras. First, the group D_8 of essentially strict spectral isomorphisms as described in Remark [54, Remark 7.7] acts on the collection of objects of $\mathfrak{C}_{\mathrm{class}}$. This corresponds to the action of D_8 on the set V by preserving d, and on a pair (m_-, m_+), the action is generated by the interchanging m_- and m_+ and by sign changes of the $m_\pm$. Observe that these operations preserve the type X.

Then there exist additional basic STMs in $\mathfrak{C}_{\mathrm{class}}$ of the types indicated below. (In these formulas we have used the notation $\epsilon(x) = x/|x| \in \{\pm 1\}$ to denote the signature of a nonzero rational number x.) In the first 5 cases one of the parameters m_- or m_+ is translated by a step of size 1 (if the translated parameter is half integral) or 2 (if the translated parameter is integral) in a direction such that its absolute value decreases. In these first 5 cases both parameters can be translated in this way, as long as the absolute value of this parameter is larger than $\frac{1}{2}$ (in the half integral case) or 1 (in the integral case). A formula corresponds to an STM provided that this condition on the absolute value of the parameter which will be translated is satisfied.

$$\begin{array}{ll}
\mathrm{C}_d(m_-, m_+)[q^2] \rightsquigarrow \mathrm{C}_{d+|m_-|-\frac{1}{2}}(m_- - \epsilon(m_-), m_+)[q^2] & \text{if } (m_-, m_+) \in V^{\mathrm{I}}, m_+ \notin \mathbb{Z} \\
\mathrm{C}_d(m_-, m_+)[q^2] \rightsquigarrow \mathrm{C}_{d+2(|m_+|-1)}(m_-, m_+ - 2\epsilon(m_+))[q^2] & \text{if } (m_-, m_+) \in V^{\mathrm{I}}, m_+ \in \mathbb{Z} \\
\mathrm{C}_d(m_-, m_+)[q] \rightsquigarrow \mathrm{C}_{d+|m_-|-\frac{1}{2}}(m_- - \epsilon(m_-), m_+)[q] & \text{if } (m_-, m_+) \in V^{\mathrm{II}} \\
\mathrm{C}_d(m_-, m_+)[q] \rightsquigarrow \mathrm{C}_{d+2(|m_+|-1)}(m_-, m_+ - 2\epsilon(m_+))[q] & \text{if } (m_-, m_+) \in V^{\mathrm{III}} \\
\mathrm{C}_d(m_-, m_+)[q] \rightsquigarrow \mathrm{C}_{d+2(|m_+|-1)}(m_-, m_+ - 2\epsilon(m_+))[q] & \text{if } (m_-, m_+) \in V^{\mathrm{IV}} \\
\mathrm{C}_d(m_-, m_+)[q^2] \rightsquigarrow \mathrm{C}_{2d+\frac{1}{2}a(a+1)+2b(b+1)}(\delta_-, \delta_+)[q] & \text{if } (m_-, m_+) \in V^{\mathrm{V}} \\
\mathrm{C}_d(m_-, m_+)[q^2] \rightsquigarrow \mathrm{C}_{2d+\frac{1}{2}a(a+1)+2b^2-\delta_+}(\delta_-, \delta_+)[q] & \text{if } (m_-, m_+) \in V^{\mathrm{VI}}.
\end{array}$$

We denote the first 5 cases of these STMs by $\Phi^{(m_-,m_+)}_{(d,-)}$ or $\Phi^{(m_-,m_+)}_{(d,+)}$, where the sign $\pm$ in the subscript indicates which of the parameters m_- or m_+ will be translated. Notice that if we combine the basic STMs of the first 5 cases with the group D_8 of spectral isomorphisms of $\mathfrak{C}_{\mathrm{class}}$, then we are allowed for all objects $X \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$ either one of m_- and m_+ (by a step of size 1 or 2 depending on the residue modulo $\mathbb{Z}$ of the parameter to be translated) as long as the absolute value of this parameter can still be reduced by such steps. Observe that these steps preserve the type X.

Finally we are of course allowed to compose these basic STMs thus obtained with each other and with the group D_8 of spectral isomorphisms. The basic translation steps as above commute with each other and have the obvious commutation relations with the group D_8 of spectral isomorphisms (this also follows easily from the essential uniqueness of STMs discussed below, see Proposition 4.1). Observe that while the parameters are strictly decreasing with these basic translation steps, the rank is strictly increasing.

Among the objects of $\mathfrak{C}^X_{\mathrm{class}}$ of the types $X \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$, the minimal spectral isogeny classes of objects (in the sense of [54, Definition 8.1]) are of the form:

$$\begin{array}{ll}
[\mathrm{C}_l(0, \tfrac{1}{2})[q^2]] \text{ and } [\mathrm{C}_l(1, \tfrac{1}{2})[q^2]] & \text{if } X = \mathrm{I}, \\
[\mathrm{C}_l(\tfrac{1}{2}, \tfrac{1}{2})[q]] & \text{if } X = \mathrm{II}, \\
[\mathrm{C}_l(0, 1)[q]] & \text{if } X = \mathrm{III}, \\
[\mathrm{C}_l(0, 0)[q]] \text{ and } [\mathrm{C}_l(1, 1)[q]] & \text{if } X = \mathrm{IV}.
\end{array}$$

Note that for all objects in $\mathfrak{C}^X_{\mathrm{class}}$, the spectral isogeny class of an object is just its isomorphism class [54, Proposition 8.3]. By abuse of language we will sometimes call the objects in a minimal (least) spectral isogeny class in this sense also "minimal" (respectively "least"). Note that some of these minimal objects admit a group of order 2 of spectral automorphisms (the cases $X = \mathrm{II}$ or IV).

The cases $X \in \{\mathrm{V}, \mathrm{VI}\}$ are of a different nature. There are no STMs between the different objects of these cases, as we will see below. But from each object of $\mathfrak{C}^{V}_{\mathrm{class}}$, there is an essentially unique (i.e., unique up to spectral automorphisms) STM to the least object in $\mathfrak{C}^{\mathrm{III}}_{\mathrm{class}}$ and from each object of $\mathfrak{C}^{V}_{\mathrm{class}}$ there is an essentially unique STM to one of the two types of minimal objects in $\mathfrak{C}^{\mathrm{IV}}_{\mathrm{class}}$. We call these STMs *extraspecial*.

It is easy to give a representing morphism $\phi = \phi^{(m_-,m_+)}_{(d,\pm)}$ defining the basic STMs of this kind. The first 4 cases, the building blocks of elementary translations in the parameters m_- and m_+, do in general not correspond to geometric diagrams as given in [40] and [43], since the image of the spectral transfer map is in general not a least object. However, as we will see below, these building blocks are quite simple and their existence can be established easily by a direct computation. The extraspecial cases correspond to the geometric diagrams [40, 7.51, 7.52] and to [43, 11.5] (in a way that will be made precise below).

The formula defining this morphim ϕ for the first 5 cases (thus a minimal translation step in one of the parameters of an object of type $X \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$) only depends on the value modulo $\mathbb{Z}$ of the parameter to be translated. Using the group D_8 of spectral isomorphisms it is enough to write down the formula for a basic translation in m_+ where $m+ > 0$.

Let $(d; (m_-, m_+)) \in V$. First, assume that $m_+ \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$. For $d \geq 0$ consider the torus $T_d(\mathbf{L}) := \mathbb{G}^r_m(\mathbf{L})$ over $\mathbf{L}$. We write its character lattice as $X^*(T_d) := X_d$ (or $X_d = \mathbb{Z}^d$). The standard basis of $X^*(T_d)$ is denoted by $(t_1, \ldots, t_d)$. We consider X_d as the root lattice of the root system of type B_d. The Weyl group W_0 acts by signed permutations on X_d. For $m_\pm \in \mathbb{Z}+1/2$ we define a homomorphism $\phi^{(m_-,m_+)}_{(d,+),T} : T_d \to T_{d+m_+-1/2}$ of algebraic tori over $\mathbf{L}$ by

$$\phi^{(m_-,m_+)}_{(d,+),T}(t_1, \ldots, t_d) := (t_1, t_2, \ldots, t_d, v^{\mathfrak{b}}, v^{3\mathfrak{b}}, \ldots, v^{2\mathfrak{b}(m_+-1)}).$$

Next, if $m_+ \in \mathbb{Z}_{>0}$, we define a morphism $\phi^{(m_-,m_+)}_{(d,+),T} : T_d \to T_{d+2(m_+-1)}$ of algebraic tori over $\mathbf{L}$ by

$$\phi^{(m_-,m_+)}_{(d,+),T}(t_1, \ldots, t_d) := (t_1, t_2, \ldots, t_d, 1, q^{\mathfrak{b}}, q^{\mathfrak{b}}, q^{2\mathfrak{b}}, q^{2\mathfrak{b}}, \ldots, q^{\mathfrak{b}(m_+-2)}, q^{\mathfrak{b}(m_+-2)}, q^{\mathfrak{b}(m_+-1)}).$$

Finally, for the extraspecial cases $X \in \{\mathrm{V}, \mathrm{VI}\}$, we define, for $m_\pm > 0$, a morphism $\phi^{(m_-,m_+)}_{(d,+),T} : T_d \to T_L$ with $L := 2d + \frac{1}{2}a(a+1) + 2b(b+1)$ (if $X = \mathrm{V}$) or $L := 2d + \frac{1}{2}a(a+1) + 2b^2 - \delta_+$ (if $X = \mathrm{VI}$) as follows. Observe that $L = 2d + \lfloor L_- \rfloor + \lfloor L_+ \rfloor$ where $L_\pm := \kappa_\pm(2\kappa_\pm + 2\epsilon_\pm - 1)/2$. We first define, for $m \in \mathbb{Z} \pm \frac{1}{4}$, residual points $r_e(m)$ recursively by putting, for $m > 1$,

$$r_e(m) = (\sigma_e(m); r_e(m-1))$$

with, for $m > 1$,

$$\sigma_e(m) = (q^{\delta}, q^{\delta+1}, \ldots, q^{2m-\frac{3}{2}}),$$

and $r_e(\frac{1}{4}) = r_e(\frac{3}{4}) := \emptyset$. We define the representing morphism of the extraspecial STM by

$$\phi^{(m_-,m_+)}_{(d,+),T}(t_1,\ldots,t_d) := (-r_e(m_-), v^{-1}t_1, vt_1, \ldots, v^{-1}t_d, vt_d, r_e(m_+)). \tag{35}$$

The proof of the fact that these formulas indeed define an STM is a straightforward computation in the cases $X \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$.

In the extraspecial case one notices first that this map for general $d \geq 0$ is induced from the cuspidal map of this kind with $d = 0$. It is easy to verify that this map is an STM, by considering the set of positive roots of the root system R_0 of type $\mathrm{B}_{2d+L_-+L_+}$ which restrict to a given simple root α_i of B_d (this process is similar to what we did in the exceptional cases), *provided* that the inducing rank 0 map is indeed a cuspidal STM. The latter can be proved by induction on $m_\pm$, using the recursive definition of $r_e(m)$ and the formula (easily obtained from (33) applied to the two cases $\{\mathrm{V}, \mathrm{VI}\}$):

$$d^\tau_{m_-,m_+} = \prod_{i=1}^{\lfloor |m_- - m_+|\rfloor} \frac{v^{2(|m_--m_+|-i)i}}{(1+q^{2(|m_--m_+|-i)})^i} \prod_{j=1}^{\lfloor |m_- + m_+|\rfloor} \frac{v^{2(|m_-+m_+|-j)j}}{(1+q^{2(|m_-+m_+|-j)})^j}. \tag{36}$$

3.2.7 Existence of enough STMs for the classical cases After having established the existence of these STMs between affine Hecke algebras of type $\mathrm{C}_n^{(1)}$, it is an easy task to prove the existence of an STM of the form $\phi : \mathcal{H}^{u,\mathfrak{s},e} \rightsquigarrow \mathcal{H}^{IM}(G)$ for all absolutely simple, quasisplit adjoint groups of classical type G and unipotent affine Hecke algebras $\mathcal{H}^{u,\mathfrak{s},e}$ of a unipotent type of an inner form of G (other than PGL_{n+1}), using covering STMs.

For $G = \mathrm{PU}_{2n}$, we have a $2:1$ semi-standard spectral covering map (see [54, 7.1.3]) of the form $\mathcal{H}^{IM}(G) = \mathrm{B}_n(2,1)[q] \rightsquigarrow \mathrm{C}_n(0, \frac{1}{2})[q^2]$ corresponding to an embedding of the right-hand side as an index two subalgebra of the left-hand side. Here the right-hand side is normalized as object in $\mathfrak{C}^{\mathrm{I}}_{\mathrm{class}}$. The representing morphism ϕ_T has kernel $\omega \in T^{IM}$, the unique nontrivial $W_0(\mathrm{B}_n)$-invariant element. We can identify ω with the nontrivial element of $X^*_{\mathrm{un}}(G) = (\Omega^\theta_C)^*$, which equals C_2 in this case. It acts as a diagram automorphism on the geometric diagram via the simple affine reflection $\sigma = s_{1-2x_1}$ of the afine Weyl group of type $\mathrm{C}_n^{(1)}$ (in the standard coordinates for $\mathfrak{t}$).

For $G = \mathrm{PU}_{2n+1}$ we have an isomorphism $\mathcal{H}^{IM}(G) \rightsquigarrow \mathrm{C}_n(\frac{1}{2}, 1)[q^2]$. These target affine Hecke algebras are the minimal objects of $\mathfrak{C}^{\mathrm{I}}_{\mathrm{class}}$. All direct summands $\mathcal{H}$ of $\mathcal{H}_{\mathrm{uni}}(G)$ either are objects of $\mathfrak{C}^{\mathrm{I}}_{\mathrm{class}}$ or, in the case $G = {}^2\mathrm{A}_{2n-1}$, otherwise there exists a semi-standard $2:1$ covering STM $\mathcal{H} \rightsquigarrow \mathcal{H}'$ arising from an index two embedding $\mathcal{H}' \subset \mathcal{H}$ of an object $\mathcal{H}'$ of $\mathfrak{C}^{\mathrm{I}}_{\mathrm{class}}$ for which one of the parameters m_- or m_+ equals 0. In the latter case, it is easy to see from the definitions that any composition ϕ_T of basic translation STMs in $\mathcal{C}^{\mathrm{I}}_{\mathrm{class}}$ which yields an STM $\mathcal{H}' \rightsquigarrow \mathrm{C}_n(0, \frac{1}{2})[q^2]$ in $\mathfrak{C}^{\mathrm{I}}_{\mathrm{class}}$ factors through an STM $\mathcal{H} \rightsquigarrow \mathcal{H}^{IM}(G)$. Let $L = rT^L$ be the image of ϕ_T. The inverse image of L under the covering map is connected if L has positive rank, this follows from the spectral map diagram and the fact that linear residual points in a positive Weyl chamber are invariant for diagram automorphisms [53]. This implies that we have a unique factorization of ϕ_T as desired in all cases. Remark that $\Omega_1^{\mathfrak{s},\theta} = 1$ except if $\mathfrak{s}$ is a supercuspidal unipotent type of G^u of type ${}^2A_{2n-1}$ or its non-quasisplit inner form, which is also Ω_C^θ-invariant. In this case $\Omega_1^{\mathfrak{s},\theta} = C_2$, and the supercuspidal STM ϕ_T as above has image L (a residual point) which lifts to two residual points which are not conjugate under $W(\mathrm{B}_n)$. Hence in this case we obtain two STMs defined by the lifts of ϕ_T, corresponding to the two summands of $\mathcal{H}^{\mathfrak{s},u}$. If $\Omega_1^{\mathfrak{s},\theta} = 1$ but $\Omega^{\mathfrak{s},\theta} = \Omega_C^\theta = C_2$, then $X_{\mathrm{un}}^*(G)$ acts nontrivially on the connected inverse image of L. This is precisely the case where one parameter of $\mathcal{H}'$ is 0, and the rank is positive. In other cases $\mathcal{H}$ is itself already an object of $\mathfrak{C}_{\mathrm{class}}^{\mathrm{I}}$ whose STM has a unique lift to $\mathcal{H}^{IM}(G)$.

For $G = \mathrm{SO}_{2n+1}$ we have $\mathcal{H}^{IM}(G) = \mathrm{C}_n(\frac{1}{2}, \frac{1}{2})[q]$, and all unipotent affine Hecke algebras are objects of $\mathfrak{C}_{\mathrm{class}}^{\mathrm{II}}$; hence this case is straightforward by the above.

For $G = \mathrm{PCSp}_{2n}$ we have a semi-standard STM $\mathcal{H}^{IM}(G) \rightsquigarrow \mathrm{C}_n(0, 1)[q]$ arising from an embedding of the right-hand side as an index two subalgebra of the left-hand side. Here the right-hand side is normalized as object in $\mathfrak{C}_{\mathrm{class}}^{\mathrm{III}}$. All direct summands of $\mathcal{H}_{\mathrm{uni}}(G)$ either are objects of $\mathfrak{C}_{\mathrm{class}}^{\mathrm{III}}$ or $\mathfrak{C}_{\mathrm{class}}^{\mathrm{V}}$, or there exists a semi-standard covering STM $\mathcal{H} \rightsquigarrow \mathcal{H}'$ arising from an index two embedding $\mathcal{H}' \subset \mathcal{H}$ of an object $\mathcal{H}'$ of $\mathfrak{C}_{\mathrm{class}}^{\mathrm{III}}$ for which one of the parameters m_- or m_+ equals 0. Similar remarks as in the case PU_{2n+1} apply on how to obtain STMs of direct summands $\mathcal{H}$ of $\mathcal{H}_{\mathrm{uni}}(G)$ to $\mathcal{H}^{IM}(G)$ in terms of those of $\mathcal{H}'$ to $\mathrm{C}_n(0, 1)[q]$.

For $G = \mathrm{P}(\mathrm{CO}_{2n}^0)$, we have a non-semistandard STM $\mathcal{H}^{IM}(G) \rightsquigarrow \mathrm{C}_n(0, 0)[q]$ which is represented by a degree 2 covering of tori (essentially the "same" covering of tori as for the case PU_{2n+1}, but this time equipped with the action of $W(\mathrm{D}_n)$ instead of $W(\mathrm{B}_n)$) (see [54, 7.1.4]). Here the right-hand side is normalized as an object in $\mathfrak{C}_{\mathrm{class}}^{\mathrm{IV}}$. All other direct summends $\mathcal{H}$ of $\mathcal{H}_{\mathrm{uni}}(G)$ either are objects of $\mathfrak{C}_{\mathrm{class}}^{\mathrm{IV}}$ with both m_- and m_+ even, or of $\mathfrak{C}_{\mathrm{class}}^{\mathrm{VI}}$ with $\delta_- = \delta_+ = 0$, or there exists a semi-standard covering STM $\mathcal{H} \rightsquigarrow \mathcal{H}'$ arising from an index two embedding $\mathcal{H}' \subset \mathcal{H}$ of an object $\mathcal{H}'$ of $\mathfrak{C}_{\mathrm{class}}^{\mathrm{IV}}$ for which one of the parameters m_- or m_+ equals 0. If both of $m_\pm \neq 0$, then $\Omega^{\mathfrak{s},\theta} = 1$, and any composition of basic STMs or the extraspecial STM $\phi : \mathcal{H} \rightsquigarrow \mathrm{C}_n(0, 0)[q]$ admits a unique lift to an STM $\phi : \mathcal{H} \rightsquigarrow \mathcal{H}^{IM}(G)$ as before. If one of $m_\pm$ equals zero, then $\Omega^{\mathfrak{s},\theta} = C_2$. As before, in the positive rank case we have $\Omega_1^{\mathfrak{s},\theta} = 1$, and $X_{\mathrm{un}}^*(G)$ acts non-trivially by spectral isomorphisms, via its quotient $(\Omega^{\mathfrak{s},\theta})^* = C_2$, on the connected inverse image of the residual coset L which is the image of ϕ. Finally if one of $m_\pm = 0$ and the rank of ϕ is 0, then $\Omega^{\mathfrak{s},\theta} = \Omega_1^{\mathfrak{s},\theta} = C_2$, and L has two lifts under the $2 : 1$ covering which are not in the same $W(\mathrm{D}_n)$-orbit but which are exchanged by the action of $X_{\mathrm{un}}^*(G)$. In this case, $\mathcal{H}^{u,\mathfrak{s}}$ decomposes as a direct sum of two copies of $\mathbf{L}$, and we have still an essentially unique STM $\mathcal{H}^{u,\mathfrak{s}} \rightsquigarrow \mathcal{H}^{IM}(G)$.

For $G = \mathrm{P}((\mathrm{CO}_{2n+2}^*)^0)$. We have $\mathcal{H}^{IM}(G) = \mathrm{C}_n(1, 1)[q]$. This case is similar to the previous case, except that this time the relevant objects from $\mathfrak{C}_{\mathrm{class}}^{\mathrm{IV}}$ are those with m_- and m_+ both odd, and those of $\mathfrak{C}_{\mathrm{class}}^{\mathrm{VI}}$, the objects with $\delta_- = \delta_+ = 1$. Hence this case is easier, since the direct summands of $\mathcal{H}$ of $\mathcal{H}_{\mathrm{uni}}(G)$ are themselves already objects of $\mathfrak{C}_{\mathrm{class}}^{\mathrm{IV}}$ and of $\mathfrak{C}_{\mathrm{class}}^{\mathrm{VI}}$, and no dicussion of lifting of STMs is required.

3.2.8 Proof of Theorem 3.4 Suppose G is as in Theorem 3.4. In the previous paragraphs we established the existence of at least one STMs $\phi_{u,\mathfrak{s},e} : \mathcal{H}^{u,\mathfrak{s},e}(G) \rightarrow \mathcal{H}^{IM}(G)$ for every unipotent type $\mathfrak{s}$ of any inner form G^u of G. Such an STM $\phi_{u,\mathfrak{s},e}$

determines a unique subset $Q \subset F_0$ such that $\phi_{u,\mathfrak{s},e}$ is represented by a morphism ϕ_T whose image is of the form $L_n := L/K_L^n$ with $L = r_Q T^Q$ and r_Q a residue point of the semisimple subquotient $\mathcal{H}_Q^{IM}$ of $\mathcal{H}^{IM}(G)$. This r_Q is determined up to the action of K_Q.

As was explained in 3.1.3., ϕ is in this situation induced from a cuspidal STM $\phi_Q : (\mathbf{L}, \tau_0) := \mathcal{H}_0 \rightsquigarrow \mathcal{H}_Q^{IM} = \mathcal{H}(M_{ssa})$. *Suppose that we know that the essential uniqueness for the cuspidal case of Theorem 3.4 holds.* Then $W_Q r_Q$ is determined by $\mathcal{H}_0$ up to the action of $\mathrm{Aut}_{\mathrm{es}}(\mathcal{H}_Q^{IM})$ (see the argument in 3.1.3.), and since we are clearly in the standard case, this is anti-isomorphic to $\Omega_{X_Q}^* \rtimes \Omega_0^{Y_Q}$ by Proposition [54, Proposition 3.4]. But we know (see [53], [52, Theorem A.14(3)]) that $W_Q r_Q$ is fixed for the action of $\Omega_0^{Y_Q}$, so that we need to consider only the orbit of $W_Q r_Q$ for the action of $\Omega_{X_Q}^* := (X_Q/\mathbb{Z}R_Q)^*$. In the case at hand $X_Q := X/X \cap R_Q^\perp = P(R_0)/P(R_0) \cap R_Q^\perp = P(R_Q)$, so that $(\Omega_{X_Q})^* = (P(R_Q)/\mathbb{Z}R_Q)^*$. But this is exactly equal to K_Q, hence any STM which is induced from a cuspidal STM $\phi_Q : (\mathbf{L}, \tau_0) := \mathcal{H}_0 \rightsquigarrow \mathcal{H}_Q^{IM}$ has as its image L_n. By the rigidity property Proposition [54, Proposition 7.13] we see that any two such STMs are equal up to the action of $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{H})$. But as was explained in 3.1.3., the subset $Q \subset F_0$ is itself completely determined by just the root system of M_{ssa}, and this is determined by $\mathfrak{s}$. It follows that any other STM $\phi' : \mathcal{H}^{u,\mathfrak{s},e}(G) \to \mathcal{H}^{IM}(G)$ can be represented by a ϕ'_T whose image is L'_n, with L' a residual coset in the $X_{\mathrm{un}}^*(G)$-orbit of L.

As to the possibility to define an equivariant STM for the action of $X_{\mathrm{un}}^*(G) = (\Omega_C^\theta)^*$, that is an application of Theorem 2.8. Recall that $X_{\mathrm{un}}^*(G)$ acts on $\mathcal{H}^{u,\mathcal{O}}$ via its quotient $(\Omega^{\mathfrak{s},\theta})^*$; we need to check in all cases that the subgroup $(\Omega_2^{\mathfrak{s},\theta})^* \subset (\Omega^{\mathfrak{s},\theta})^*$ is the stabilizer of $W_0(L)$. For the classical cases this was discussed in the previous sections, and for the exceptional cases this is an easy verification. It follows that the direct sum $\mathcal{H}^{u,\mathfrak{s}}$ of all summands of $\mathcal{H}_{\mathrm{uni}}(G)$ in the $X_{\mathrm{un}}^*(G)$-orbit of $\mathcal{H}^{u,\mathfrak{s},e}$ can be mapped $X_{\mathrm{un}}^*(G)$-equivariantly by an STM to $\mathcal{H}^{IM}(G)$, and that such an equivariant STM is essentially unique up to the spectral automorphism group of $\mathcal{H}_{\mathrm{uni}}(G)$. Taking the direct sum over all orbits $X_{\mathrm{un}}^*(G)$-orbits of unipotent types, we obtain the desired result.

Hence Theorem 3.4 is now reduced to the cuspidal case. For the exceptional cases we have already shown the essential uniqueness for cuspidal STMs, and for $G = \mathrm{PGL}_{n+1}$ this was obvious. Hence the Proof of Theorem 3.4 is completed by the following result, whose proof will appear in [19]:

Proposition 3.8 [19] *The essential uniqueness of Theorem 3.4 holds true for the cuspidal part (or rank 0 part) $\mathcal{H}_{uni,cusp}(G)$ of $\mathcal{H}_{uni}(G)$ for G of type PU_n, SO_{2n+1}, $PCSp_{2n}$, $P(CO_{2n}^0)$, and $P((CO_{2n}^*)^0)$. Here we denote by $\mathcal{H}_{uni,cusp}(G)$ the direct sum of all the cuspidal (or rank 0) normalized generic unipotent affine Hecke algebras associated to G and its inner forms. In other words, there do not exist other rank 0 STMs than the ones constructed above, and this yields a $X_{un}^*(G)$-equivariant STM $\Phi_{cusp} : (\mathcal{H}_{uni,cusp}(G), \tau) \rightsquigarrow (\mathcal{H}^{IM}, \tau^{IM})$ which is* essentially unique *in the sense of Theorem 3.4.*

The proof of this proposition reduces to the analogous statement for the spectral categories $\mathfrak{C}_{\mathrm{class}}^X$. For $X = \mathrm{I}, \mathrm{II}$ this is rather easy. When $X = \mathrm{III}, \mathrm{IV}, \mathrm{V}, \mathrm{VI}$ the *essential uniqueness* proof for cuspidal STMs is based on the *existence* of the cuspidal extraspecial STMs of $\mathfrak{C}_{\text{class}}^{\text{V}}$ and $\mathfrak{C}_{\text{class}}^{\text{VI}}$. It is easy to see that every generic residual point of $\mathfrak{C}_{\text{class}}^{X}$ for $X = \text{III, IV}$ is in the image of a unique extraspecial cuspidal STM, and this sets up a bijection between the set of generic residual points of the combined objects of $\mathfrak{C}_{\text{class}}^{\text{V, VI}}$ and those of $\mathfrak{C}_{\text{class}}^{\text{III, IV}}$. If we impose the necessary condition for cuspidality, namely that the formal degree (in our normalization) has no odd cyclotomic factors, then one can show that the corresponding generic residual point of $\mathfrak{C}_{\text{class}}^{\text{V, VI}}$ is given by a pair (ξ_-, ξ_+) of partitions whose Young tableaux are of rectangular shape, and almost a square. After applying the extraspecial STM, the solutions correspond to a pair (u_-, u_+) of unipotent orbits of $G_s \subset G$, a semisimple subgroup of maximal rank, whose elementary divisors are both of the form $(1, 3, 5, \dots)$ or $(2, 4, 6, \dots)$, or are both of the form $(1, 5, 9, \dots)$ or $(3, 7, 11, \dots)$. These solutions thus correspond to the cuspidal local systems for the endoscopic groups $G_s \subset G$ (cf. [34,37]).

4 Applications

4.1 Classification of unipotent spectral transfer morphisms

4.1.1 The classical case

Proposition 4.1 *Between the objects of $\mathfrak{C}_{class}^{I}$, $\mathfrak{C}_{class}^{II}$, $\mathfrak{C}_{class}^{III\cup V}$ and $\mathfrak{C}_{class}^{IV\cup VI}$ (where $\mathfrak{C}_{class}^{III\cup V}$ is shorthand for $\mathfrak{C}_{class}^{III} \cup \mathfrak{C}_{class}^{V}$ etc.) all STMs are generated by the basic translation STMs we defined in 3.2.7, the extraspecial STMs, and the dihedral group D_8 (cf.* [54, Remark 7.5]*) of spectral isomorphisms. The basic translation STMs commute with each other, and the commutation rules of the basic translation STMs and extra special STMs with the D_8 are the obvious ones, where D_8 acts on the set of parameter pairs (m_-, m_+) (i.e., D_8 acts as a group of endofunctors on each of these categories).*

Proof For any object $\mathcal{H}$ in $\mathfrak{C}_{\text{class}}^{Y}$ (Y as in the Theorem) there exists an STM ϕ : $\mathcal{H} \rightsquigarrow \mathcal{H}^{\text{min}}$, where $\mathcal{H}^{\text{min}}$ denotes a minimal object, and where ϕ is a translation STM or an extraspecial STM. By the essential uniqueness of Theorem 3.4 it follows that any STM $\psi : \mathcal{H} \rightsquigarrow \mathcal{H}^{\text{min}}$ is of the form $\psi = \beta \circ \phi \circ \alpha$ with $\alpha \in \text{Aut}_{\text{es}}(\mathcal{H})$ and with $\beta \in \text{Aut}_{\text{es}}(\mathcal{H}^{\text{min}})$. In $\mathfrak{C}_{\text{class}}^{Y}$, the group $\text{Aut}_{\text{es}}(\mathcal{H})$ is trivial (if the parameters m_- and m_+ are unequal) or C_2 (if the parameters are equal). If there exists a nontrivial $\alpha_0 \in \text{Aut}_{\text{es}}(\mathcal{H})$, then $m_- = m_+$ and it follows easily from the definitions that there also exists a nontrivial $\beta_0 \in \text{Aut}_{\text{es}}(\mathcal{H}^{\text{min}})$, and that ϕ is equivariant in the sense $\phi \circ \alpha_0 = \beta_0 \circ \phi$. Hence, if ψ is also a composition of basic translation STMs or if ψ is an extra special STM, we see that $\psi = \phi$. From the injectivity (obvious from the definitions) of the basic generating STMs it now follows that the basic translation STMs commute.

We also conclude from the injectivity of the basic generating STMs that, up to spectral isomorphisms, there can exist at most one STM between any two objects of $\mathfrak{C}_{\text{class}}^{Y}$. For $X \in \{\text{I, II, III, IV}\}$ it follows from a consideration of the spectral transfer map diagrams (Definition 3.3) of the (essentially unique, injective) STMs $\phi_1 : \mathcal{H}_1 \rightsquigarrow \mathcal{H}^{\text{min}}$ and $\phi_2 : \mathcal{H}_2 \rightsquigarrow \mathcal{H}^{\text{min}}$ that a possible factorizing STM $\phi : \mathcal{H}_1 \rightsquigarrow \mathcal{H}_2$ (uniquely determined if it exists) must be itself composed of basic translation STMs and spectral isomorphism itself. It is also easy to see in this way that there can not exist STMs between objects of $\mathfrak{C}_{\text{class}}^{\text{V}\cup\text{VI}}$ and non-minimal objects of $\mathfrak{C}_{\text{class}}^{X}$ with $X \in \{\text{I, II, III, IV}\}$.

Between objects of $\mathfrak{C}_{\text{class}}^{X}$ for $X \in \{\text{V, VI}\}$ there are no STMs. This again follows from the injectivity of the extra special STMs, in view of the fact that the images of two extraspecial STMs of the form $\phi_1 : \mathcal{H}_1 \rightsquigarrow \mathcal{H}^{\min}$ and $\phi_2 : \mathcal{H}_2 \rightsquigarrow \mathcal{H}^{\min}$ map to disjoint subsets of the spectrum of the center of $\mathcal{H}^{\min}$, unless $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic (this follows from the "extra special bijection" proved in [19]). □

As a consequence we obtain a general description of all spectral transfer maps between all unipotent affine Hecke algebras in the classical cases:

Corollary 4.2 *Let* **G** *be connected, absolutely simple, defined and quasisplit over k, split over K, and such that its restricted root system is of classical type. There are no other STMs between the unipotent affine Hecke algebras of the form $\mathcal{H}^{u,\mathfrak{s},e}$ which appear as summands of $\mathcal{H}_{uni}(G)$ than the ones obtained by lifting STMs via spectral covering maps of direct summands of $\mathcal{H}_{uni}(G)$ to one of $\mathfrak{C}_{class}^{I}$, $\mathfrak{C}_{class}^{II}$, $\mathfrak{C}_{class}^{III\cup V}$ and $\mathfrak{C}_{class}^{III\cup V}$ (lifting in a sense similar to the discussion in 3.2.7).*

It is not difficult to describe all STMs between the unipotent affine Hecke algebras for exceptional types as well, but we will not do this here.

4.2 Partitioning of unramified square integrable L-packets according to Bernstein components

Let **G** be connected, absolutely simple, defined and quasisplit over k, split over K, and of adjoint type. Let $\mathcal{H}'$ be a unipotent affine Hecke algebra associated to a unipotent type of an inner form of G (hence, a summand of $\mathcal{H}_{\text{uni}}(G)$). By our Theorem 3.4 we know that there exists an essentially unique $X_{\text{un}}^{*}G$-equivariant STM $\phi : \mathcal{H}_{\text{uni}}(G) \rightsquigarrow \mathcal{H}^{IM}(G)$, and we know that such a map is compatible with the arithmetic/geometric correspondence of diagrams of Lusztig [40,43]. By [54, Theorem 6.1], this STM ϕ gives rise to a correspondence between components of the tempered irreducible spectra of $\mathcal{H}'$ and $\mathcal{H}^{IM}(G)$ which preserves, up to rational constant factors, the Plancherel densities on these components, and which is compatible with the map ϕ_Z on the level of central characters of representations. In particular for unipotent discrete series representations, given an orbit of residual points $W_0 r_{\mathbf{L}} \in W_0 \backslash T(\mathbf{L})$ for $\mathcal{H}^{IM}(G)$ (these carry the discrete series representations, by [52]), we collect the irreducible discrete series characters of the various direct summands $\mathcal{H}' = \mathcal{H}^{u,\mathfrak{s},e}$ of $\mathcal{H}_{\text{uni}}(G)$ whose central character $W_0' r_{\mathbf{L}}'$ satisfies $\phi_Z(W_0' r_{\mathbf{L}}') = W_0 r_{\mathbf{L}}$.

Definition 4.3 Given an orbit $W_0 r_{\mathbf{L}}$ of **L**-residual points of $\mathcal{H}^{IM}(G)$ we form a packet $\Pi_{W_0 r_{\mathbf{L}}}$ consisting of the unipotent discrete series characters of inner forms of G for which the corresponding discrete series representation of $\mathcal{H}'$ (the corresponding summand of $\mathcal{H}_{\text{uni}}(G)$) has a central character $W_0' r_{\mathbf{L}}'$ which satisfies $\phi_Z(W_0' r_{\mathbf{L}}') = W_0 r_{\mathbf{L}}$ (with ϕ_Z as above).

Corollary 4.4 *By Theorem* [54, Theorem 6.1]*, the q-rational part of the formal degree of all the irreducible characters in $\Pi_{W_0 r_{\mathbf{L}}}$ is the same.*

There exists a natural bijection (cf. [52, Corollary B.5], and paragraph 2.3) $\Lambda^e \ni [\lambda] \to W_0 r_{\lambda,\mathbf{L}}$ between orbits of discrete unramified Langlands parameters and orbits of residual points $W_0 r_{\mathbf{L}}$ for $\mathcal{H}^{IM}$. Theorem 3.4 implies that the packets $\Pi_{W_0 r_{\lambda,\mathbf{L}}}$ defined by STMs, admit a classification in terms of local systems on the $G^\vee$-orbits of discrete unramified Langlands parameters:

Corollary 4.5 ([40–43, Theorem 5.21]) *The packet $\Pi_{W_0 r_{\lambda,\mathbf{L}}}$ can be parameterized by the disjoint union of the fibres $\tilde{\Lambda}^u_\lambda$ (cf. paragraph 2.3 for this notation), where $u \in N_G(\mathbb{B})$ corresponds to the various inner forms of G via Kottwitz's Theorem (here we identify $N_G(\mathbb{B})$ with the character group $\Omega/(1-\theta)\Omega)$ of the center ${}^L Z$ of ${}^L G$ (cf. Sect. 2.3).*

In [13,56] the discrete series characters of arbitrary affine Hecke algebra $\mathcal{H}$ are parameterized differently. This point of view will be quite fruitful for the applications we have in mind, especially for unequal parameter Hecke algebras, and this is what we will discuss next.

Let $\mathbb{L}$ be a the ring of complex Laurent polynomials over the natural maximal algebraic torus of (possibly unequal) Hecke parameters associated to the underlying root datum of $\mathcal{H}$ (this ring was denoted by Λ in [56]). Explicitly, $\mathbb{L}$ is the ring of Laurent polynomials in invertible indeterminates $v_{\alpha,\pm}$ (with $\alpha \in R_0$) subject to the conditions $v_{\alpha,\pm} = v_{w(\alpha),\pm}$ for all $\alpha \in R_0$ and $w \in W_0$, and $v_{\alpha,-} = 1$ iff $1-\alpha^\vee \in W\alpha^\vee$). We give $\mathbf{L}$ the structure of a $\mathbb{L}$-algebra by putting $v_{\alpha,\pm} = v^{m_\pm(\alpha)}$. Then we have a generic affine Hecke algebra $\mathcal{H}_{\mathbb{L}}$ defined over $\mathbb{L}$, and $\mathcal{H} = \mathbf{L} \otimes_{\mathbb{L}} \mathcal{H}_{\mathbb{L}}$.

Let $\mathbb{V}$ be the space of points of the maximal spectrum of $\mathbb{L}$ such that for all $\underline{\mathbf{v}} = (\mathbf{v}_{\alpha,\pm}) \in \mathbb{V}$, we have $\mathbf{v}_{\alpha,\pm} := \underline{\mathbf{v}}(v_{\alpha_\pm}) \in \mathbb{R}_+$ for all $\alpha \in R_0$. Let $\mathbf{V}$ be the space of points $\mathbf{v} \in \mathbb{R}+$ of the maximal spectrum of $\mathbf{L}$. Thus we have an embedding $\mathbf{V} \hookrightarrow \mathbb{V}$, and $\mathbf{v}_{\alpha,\pm} = \mathbf{v}^{m_\pm(\alpha)}$.

It was shown in [56, Theorem 3.4, Theorem 3.5] that an irreducible discrete series character δ of $\mathcal{H}$ is the specialization $\delta = \widetilde{\delta}_{\mathbf{L}}$ at $\mathbf{L}$ of a *generic family* of irreducible discrete series characters $\widetilde{\delta}$ of $\mathcal{H}_{\mathbb{L}}$ which is well defined in an open neighborhood of $(\mathbf{v}^{m_\pm(\alpha)})$. Each discrete series character of $\mathbb{H}$ can thus be locally deformed in the parameters $m_\pm(\alpha)$. We will write such deformation as $m^\epsilon_\pm(\alpha) = m_\pm(\alpha) + \epsilon_\pm(\alpha)$, where $\epsilon_\pm(\alpha)$ vary in a sufficiently small open interval $(-\epsilon, \epsilon) \subset \mathbb{R}$. The irreducible discrete series representations of affine Hecke algebras with arbitrary positive parameters $(\mathbf{v}^{m^\epsilon_\pm(\alpha)})$, so in particular of all affine Hecke algebras of the form $\mathcal{H}' = \mathcal{H}^{u,\mathfrak{s},e}$, have been classified in [56] from the point of view of deformations over the ring $\mathbb{L}$.

In the case of nonsimply laced irreducible root systems $R_0^{u,\mathfrak{s},e}$, the classification of [56] is in terms of the *generic central character map gcc* which associates to any irreducible discrete series character a $W_0 := W(R_0^{u,\mathfrak{s},e})$-orbit $W_0 r$ of *generic residual points*. A generic residual point $r \in T(\mathbb{L})$ is an $\mathbb{L}$-valued point where μ has maximal pole order. The set of such points is finite and invariant for the action of W_0.

We can choose the generic residual point r always of the form (see [56, Theorem 8.7]) $r = s(e)\exp(\xi) \in T(\mathbb{L})$, where e runs over s complete set of representatives of the $\Gamma := Y/Q(R_1^\vee)$-orbits of vertices of the spectral diagram $\Sigma_s(\mathcal{R}^m)$, and $s(e)$ is the corresponding vertex of the dual fundamental alcove $C^\vee \subset \mathbb{R} \otimes Y$. This gives rise to a semisimple subroot system $R_{s(e),1} \subset R_1$, with the parameter function $m_{\pm}^{\epsilon,e}(\alpha)$ obtained by restriction of the parameters $m_{\pm}^{\epsilon}(\alpha)$ to the subdiagram of the geometric diagram $\Sigma_s(\mathcal{R}^m)$ obtained by omitting the vertex e, and replacing the group of diagram automorphisms Γ by the isotropy subgroup $\Gamma_e \subset \Gamma$. Finally, ξ denotes a *linear residual point* (see [56, Section 6]) for the generic graded affine Hecke algebra defined by $R_{s(e),1}$ and $m_{\pm}^{\epsilon,e}(\alpha)$.

Thus ξ depends linearly on parameters $m_{\pm}^{\epsilon,e}(\alpha)$ of the graded affine Hecke algebra. The specialization W_0r_0 of the orbit W_0r at $\epsilon_{\pm} = 0$ is a confluence of finitely many orbits of generic residual points W_0r_i, with $i \in I_{W_0r_0}$ (some finite set which one can explicitly determine, see [56, Section 6]) from the explicit classification of linear residual points). For each irreducible discrete series character δ with central character W_0r_0, its unique continuous deformation $\tilde{\delta}$, locally in the Hecke parameters, has a central character of $\tilde{\delta}$ equal to one of the orbits W_0r_i of generic residual points which specialize at $\epsilon_{\pm} = 0$ to W_0r_0.

This defines [56] a unique "generic central character" map gcc" from the set of irreducible discrete series at central character W_0r_0 to the set $I_{W_0r_0}$ turns out to be bijective with the single exception of the orbit of generic residual points denoted f_8 of F_4 (which is one of the three generic residual points which come together at the weighted Dynkin diagram of the minimal unipotent orbit of F_4). In this case, there are *two* generic discrete series associated to f_8. The map gcc also works well for the affine Hecke algebras of type D_n, by relating this case with affine Hecke algebras of type $\mathrm{C}_n(0,0)[q]$. We refer to [56] for details. The cases of type E_n have to be treated in a different way (classically as in [28], or see [13]).

We would like to match up these two ways of parameterizing the discrete series characters in the packet Π_λ (with $[\lambda] \in \Lambda^e$). This will be important for the purpose of proving Theorem 4.11. Indeed, recall that the formal degree of $\tilde{\delta}$ was shown to be continuous in terms of $(\epsilon_{\pm}(\alpha))$ [56, Theorem 2.60, Theorem 5.12], and that it was given explicitly by the product formula [56, Theorem 5.12]. In addition it is known [13] that the formal degree of a generic family of discrete series characters is a product of an explicitly known rational constant and an explicit rational function of the parameters $v_{\alpha,\pm}$. This enables us to compute the rational constants in the formal degree of any discrete series character δ of any normalized unipotent affine Hecke algebra $\mathcal{H}' = \mathcal{H}^{u,\mathfrak{s},e}$ by a limit argument, using the generic family $\tilde{\delta}$ and its formal degree. Motivated by this, let us consider in more detail our parameterization with this comparison in mind.

4.3 Parameterization for classical types

For PGL_n this was discussed in paragraph 3.2.2.

For classical groups (other than type A) everything is governed by Hecke algebras of the form $\mathrm{C}_n(m_-, m_+)[q^{\mathfrak{b}}]$, via the spectral correspondences of certain spectral covering morphisms. These correspondences can be made explicit by restriction and induction operations with respect to subalgebras of equal rank, and this will be discussed in detail when treating the various cases of classical type. In this paragraph we will concentrate on the principles for Hecke algebras of type $\mathrm{C}_n(m_-, m_+)[q^{\mathfrak{b}}]$.

The corresponding graded affine Hecke algebras have a root system of type $R_{s(e),0}$ of type $\mathrm{B}_{n_-} \times \mathrm{B}_{n+}$, where $n_- + n_+ = n$, and graded Hecke algebra parameters (m_-, m_+). A $W_{s(e),0}$-orbit of generic linear residual points is given by the $W_{s(e),0} = W_0(\mathrm{B}_{n_-}) \times W_0(\mathrm{B}_{n_+})$-orbit of an ordered pair (ξ_-, ξ_+), where $\xi_\pm$ is a vector of affine linear functions of $\epsilon_\pm$ such that $\xi_\pm(\epsilon_\pm)$ is the vector of contents of the boxes of the "$m_\pm^{\pm\epsilon}$-tableau" of a partition $\pi_\pm$ of $n_\pm$ with the property that at $\epsilon_\pm = 0$, the extremities [63] of the resulting $m_\pm$-tableau are all distinct.

At the "special" parameter value $m_\pm$ (integral or half-integral), the $W_0(\mathrm{B}_{n_\pm})$-orbit of the vector $\xi_\pm(0)$ is an orbit of linear residual points at parameter $m_\pm$. By a result of Slooten [63] (also see [56]), the set of such orbits of linear residual points is in bijection with the set of "unipotent partitions" $u_\pm = u_{\xi_\pm}$ of length $l_\pm \geq m_\pm - \frac{1}{2}$ of $N_\pm := 2n_\pm + (m_\pm - \frac{1}{2})(m_\pm + \frac{1}{2})$, consisting of distinct even parts (if $m_\pm$ is half integral), or a partition $u_\pm$ of length $l_\pm \geq m_\pm$ of $N_\pm := 2n_\pm + m_\pm^2$, having distinct odd parts (if $m_\pm$ is integral). Let us call such a pair $u = (u_-, u_+)$ of partitions a *distinguished unipotent partition of type* $m = (m_-, m_+)$. This set of partitions $\pi_\pm$ (and thus the set of $W_0(\mathrm{B}_{n_\pm})$-orbits of *generic* residual points $W_0(\mathrm{B}_{n_\pm})\xi_\pm$ which are confluent at $\epsilon_\pm = 0$ to the same orbit $W_0(\mathrm{B}_{n_\pm})\xi_\pm(0))$ was parameterized by Slooten in terms of the so-called $m_\pm$-symbols $\sigma_\pm$. These symbols are certain Lusztig–Shoji symbols with defect $D_\pm := \lceil m_\pm \rceil$ (see [63], [56, Definition 6.9]). Slooten's symbols [56, Definition 6.11] attached to orbits $W_0(\mathrm{B}_{n_\pm})\xi_\pm(0)$ all have the same parts, but they are distinguished from each other by the selection of the parts which appear in the top row.

Remark 4.6 In particular, there exists $\binom{2l+d}{l}$ such symbols, except when $u_\pm$ contains 0 as a part (which may happen if $m_\pm$ is half integral), in which case there are $\binom{2l+d-1}{l}$ such symbols (since 0 must appear in the top row in such case).

Let us call these Slooten's symbols associated to $u_\pm$ at parameter ratio $m_\pm$ the $u_\pm$-symbols of type $m_\pm$. The point of view in [63] is that of the deformation picture sketched above: The symbols are "confluence data", and each such symbol represents an orbit of generic residual points which evaluates to $W_0(\mathrm{B}_{n_\pm})\xi_\pm(0)$ at the parameter ratio $m_\pm$. It is convenient to formulate the results of this "abstract" classification in terms of abstract packets of representations associated to central characters of the discrete series of the minimal objects of $\mathfrak{C}^{\mathrm{I}}_{\mathrm{class}}$, $\mathfrak{C}^{\mathrm{II}}_{\mathrm{class}}$, $\mathfrak{C}^{\mathrm{III}\cup\mathrm{V}}_{\mathrm{class}}$ and $\mathfrak{C}^{\mathrm{IV}\cup\mathrm{VI}}_{\mathrm{class}}$:

Proposition 4.7 *Let $\mathcal{H}' = C_n(m_-, m_+)[q^{\mathfrak{b}}]$ be an affine Hecke algebra which appears as an object of $\mathfrak{C}^X_{class}$ for $X = I, II, III$ or IV. Let $u = (u_-, u_+)$ be an ordered pair of distinguished unipotent partitions corresponding to a central character W_0r' of a discrete series character of $\mathcal{H}'$, with $u_\pm$ of type $m_\pm$. Let $\phi : \mathcal{H}' \rightsquigarrow \mathcal{H}$ be the translation STM to a minimal object $\mathcal{H}$ of $\mathcal{C}^X_{class}$. Let $\phi_Z(W_0r') = W_0r$. Then the ordered pair of distinguished unipotent partitions corresponding to W_0r is equal to u as well! The set of irreducible discrete series characters of $\mathcal{H}'$ in $\Pi_{W_0r'}$ is parameterized by ordered pairs (σ_-, σ_+), where $\sigma_\pm$ is a $u_\pm$-symbol of type $m_\pm$. Let $\Pi^Y_{W_0r}$ be the disjoint union of all sets of irreducible discrete series characters of the objects of $\mathfrak{C}^Y_{class}$, with $Y = I, II, III \cup V$ or $IV \cup VI$, which are assigned to W_0r in this way via the translation STMs. The extraspecial STM's contribute* 1, 2 *or* 4 *elements to $\Pi^Y_{W_0r}$ (see Proposition 4.8), for each discrete series central character W_0r of $\mathcal{H}$.*

In the context of an unramified classical group of adjoint type G, the Hecke algebras of the form $\mathcal{H}^{u,\mathfrak{s}}$ are direct sums of normalized extended affine Hecke algebras $\mathcal{H}^{u,\mathfrak{s},e}$ which are spectral coverings of objects of $\mathfrak{C}_{\mathrm{class}}$. In particular, an unramified discrete Langlands parameter λ for G determines (via the comparison of the Kazhdan–Lusztig classification and the classification of discrete series representations as in [56]) an orbit of residual points W_0r for $\mathcal{H}^{IM}(G)$. In turn, via the morphism [54, Corollary 5.5] associated to this spectral covering map, this determines a pair (u_-, u_+) of distinguished unipotent partitions in the sense of Proposition 4.7, for an appropriate pair of parameters (m_-, m_+) of the form $(m_-, m_+) = (0, \frac{1}{2}),\ (1, \frac{1}{2}),\ (\frac{1}{2}, \frac{1}{2}),\ (0, 0),\ (0, 1)$ or $(1, 1)$ (see paragraph 3.2.6). Working this out amounts to determining the multiplicities and types of the normalized extended affine Hecke algebras $\mathcal{H}^{u,\mathfrak{s},e}$, and the branching rules for the algebra inclusions associated to the spectral covering maps to the relevant objects of $\mathfrak{C}_{\mathrm{class}}$. This is not difficult, and we can check that in all classical cases the STM Φ of Theorem 3.4 gives rise to packets Π_{W_0r} of discrete series characters whose members are parameterized by pairs of Slooten's symbols or come from an extraspecial STM (see paragraphs 4.6.1, 4.6.2, 4.6.3 for more details).

Slooten's symbols are known to correspond with Lusztig's symbols [37] if one uses Lusztig's arithmetic/geometric correspondences for the "geometric" graded affine Hecke algebras in the following sense. Let the central character W_0r_0 of a discrete series character π of $\mathrm{C}_n(m_-, m_+)[q^{\mathfrak{b}}]$ be given by the pair of unipotent partitions (u_-, u_+) (where $u_\pm$ has at least $\lfloor m_\pm \rfloor$ parts). The set of discrete series characters with central character W_0r_0 is parameterized by the set of generic central characters W_0r (see [56]) which evaluate to W_0r_0, via the map gcc. In turn, these generic central characters are parameterized by pairs $\sigma_-, \sigma_+)$ of the Slooten symbols (with defects $D_\pm = \lceil m_\pm \rceil$) covering (u_-, u_+). By the results of [10,42], [27, Section 4], the top graded part with respect to Slooten's functions $a_{m_\pm}$ [63] of the corresponding graded Hecke algebra module is the irreducible $W(\mathrm{C}_{n_-}) \times W(\mathrm{C}_{n_+})$-module corresponding to (σ_-, σ_+), via the generalized Springer correspondence of [46]. Via Proposition 4.7, the spectral correspondences of the standard STMs to $\mathcal{H}^{IM}(G)$ together exhaust the set of pairs of Slooten symbols (σ_-, σ_+).

The same is known to be true for the additional contributions to packet $\Pi^{Y}_{W_0r}$ coming from the extraspecial STMs (see [10], [27, Section 4]). These remarkable facts should be considered as an aspect of Langlands duality. Slooten's symbols are defined entirely in terms of affine Hecke algebras (describing the set of orbits of generic residual points specializing to the central character of a discrete series representation), whereas Lusztig's symbols describe cuspidal local systems on an associated nilpotent orbit of ${}^LG$. Comparing this with Theorem 3.4 and Proposition 4.7 we see that our parametrization of Π_{W_0r} matches with Lusztig's assignment [40,43] of unramified Langlands parameters to the members of Π_{W_0r}.

We see that the defect (D_-, D_+) of an unordered pair (σ_-, σ_+) of u-symbols for a member of Π_{W_0r} (corresponding to a pair of distinguished unipotent partitions (u_-, u_+)) determines the parameters of the affine Hecke algebra from which it originates under the STM Φ. This determines the Bernstein component to which the corresponding discrete series character belongs (up to the action of $X^*_{\mathrm{un}}(G)$).

The final statement of Proposition 4.7 will be proved in [19]. The additional contributions from the extraspecial STMs to the packets of unipotent discrete series of

PCSp_{2n}, $\mathrm{P}(\mathrm{CO}_{2n}^0)$ and $\mathrm{P}((\mathrm{CO}_{2n}^*)^0)$ correspond to the fact that one takes the centralizers of the discrete Langlands parameter in the Spin group. This gives rise to a nontrivial central extension by C_2 of the centralizer in SO_{2n} (or SO_{2n+1} respectively). These can be be described in detail in terms of central products of groups of type D_8 (the dihedral group with 8 elements), Q_8, C_2^2 or C_4 (see [37]) and among those groups we typically find extraspecial 2-groups. The precise type of the groups that arise is complicated, but we are merely interested the the number of their irreducible representations and their dimensions which is less difficult, following the description in [37] (and also using [61] for the twisted cases) one obtains:

Proposition 4.8 *Let λ be an unramified Langlands parameter for the discrete series for $PCSp_{2n}$ (with $n \geq 2$), $P(CO_{2n}^0)$ (with $n \geq 4$), and $P((CO_{2n}^*)^0)$ (with $n \geq 4$). Then λ determines an ordered pair (u_-, u_+) of distinguished unipotent partitions for the parameters $m = (m_-, m_+) = (0, 1)$ (if $G = PCSp_{2n}$), for $m = (0, 0)$ (if $G = P(CO_{2n}^0)$) or for $m = (1, 1)$ (if $G = P((CO_{2n}^*)^0)$). Let $l = (l_-, l_+)$, with $l_\pm$ the number of parts of $u_\pm$. Thus $u_\pm$ is a partition with distinct odd parts, and $|u| := |u_-| + |u_+| = 2n$ with $l_\pm$ both even if $G = P(CO_{2n}^0)$; $|u| = 2n + 1$ with l_- even and l_+ odd if $G = PCSp_{2n}$; and $|u| = 2n$ with $l_\pm$ both odd if $G = P((CO_{2n}^*)^0)$. Let us write $2^{(2a)+b}$ for a 2-group of size 2^{2a+b} which has 2^{2a+b-1} one-dimensional irreducible representations, and 2^{b-1} irreducibles of dimension 2^a.*

If u_- is the zero partition, then A_λ (as defined in paragraph 2.3) is of type $2^{(l_+-1)+1}$ if l_+ is odd, and of type $2^{(l_+-2)+2}$ if l_+ is even. If u_+ and u_- are both nonzero, then A_λ is of type $2^{(l_-+l_+-4)+3}$ if $l_\pm$ are both even, A_λ is of type $2^{(l_-+l_+-3)+2}$ if $l_\pm$ are unequal modulo 2, and A_λ is of type $2^{(l_-+l_+-2)+1}$ if $l_\pm$ are both odd.

4.4 Parameterization for split exceptional groups

For split exceptional groups, the major work to match up the irreducible discrete series characters of affine Hecke algebra summands of $\mathcal{H}_{\mathrm{uni}}(G)$ with Lusztig's parameters has been done by Reeder in [60] by computing the W-types explicitly. With this parameterization, the main Theorem of [60] is known to be a special case of the conjecture [26, Conjecture 1.4] (as discussed loc. cit.), which takes a lot of work out of our hands.

For the types E_6 and E_7 we need in addition to discuss the contribution of the nontrivial inner forms, which we take up in the next two paragraphs.

4.4.1 Inner forms of the split adjoint group G of type E_6 The inner forms of G are parameterized by $u \in \Omega \approx C_3$. We have $X_{\mathrm{un}}^*(G) = \Omega^*$. For $u = 1$ we have the following $X_{\mathrm{un}}^*(G)$ orbits of unipotent types: $\mathfrak{s}_{\emptyset}^1, \mathfrak{s}_{\mathrm{D}_4}^1, \mathfrak{s}_{\mathrm{E}_6[\theta]}^1, \mathfrak{s}_{\mathrm{E}_6[\theta^2]}^1$. For $u \neq 1$ we have the following orbits of unipotent types: $\mathfrak{s}_{\emptyset}^u, \mathfrak{s}_{{}^3\mathrm{D}_4[1]}^u, \mathfrak{s}_{{}^3\mathrm{D}_4[-1]}^u$. The orbit of $\mathfrak{s}$ is a torsor for $(\Omega_1^{\mathfrak{s}})^*$ (a quotient of $X_{\mathrm{un}}^*(G)$). By inspection we check:

Remark 4.9 In all the cases above, we have $\Omega_1^{\mathfrak{s}} = \langle u \rangle := \Omega_u \subset \Omega$.

We *choose* an equivariant bijection $\alpha \to \mathfrak{s}_\alpha$ between Ω_u^* and the orbit of $\mathfrak{s}$. Then $\mathcal{H}_{\mathrm{uni}}(G)$ is isomorphic to the direct sum of the extended affine Hecke algebras $\mathcal{H}^{u,\mathfrak{s}_\alpha,e}$

Table 1 The packets Π^u_λ for type E_6 and the contributing STMs

λ	Ω^*_λ	A_λ	STMs for Π^u_λ
E_6	1	C_3	$u = 1 : \Phi^{1,1}_{\emptyset}(E_6)$ $u \neq 1 : \Phi^{u,1}_{\emptyset}(g_1)$
$E_6(a_1)$	1	C_3	$u = 1 : \Phi^{1,1}_{\emptyset}(E_6(a_1))$ $u \neq 1 : \Phi^{u,1}_{\emptyset}(g_2)$
$E_6(a_3)$	1	$S_2 \times C_3$	$u = 1 : \Phi^{1,1}_{\emptyset}(E_6(a_3))$ $u \neq 1 : \Phi^{u,1}_{\emptyset}(g_3)$; $\Phi^{u,1}_{^3D_4[1]}$
A_1A_5	1	$C_2 \times C_3$	$u = 1 : \Phi^{1,1}_{\emptyset}(A_1A_5)$; $\Phi^{1,1}_{D_4}(A_2)$ $u \neq 1 : \Phi^{u,1}_{\emptyset}(A_1^2)$; $\Phi^{u,1}_{^3D_4[-1]}$
A_2^3	C_3	$C_3 \times C_3$	$u = 1 : \Phi^{1,1}_{\emptyset}$; $\Phi^{1,1}_{E_6[\theta]}$; $\Phi^{1,1}_{E_6[\theta^2]}$ $u \neq 1 : \Phi^{u,\alpha}_{\emptyset}(A_2)$ $(\alpha \in \Omega^*)$

where $u \in \Omega$, $\mathfrak{s}$ runs over the orbits of unipotent types, and $\alpha \in \Omega^*_u$. By Theorem 3.4 there exists an essentially unique Ω^*-equivariant collection of STMs $\Phi^{u,\alpha}_{\mathfrak{s}} : \mathcal{H}^{u,\mathfrak{s}_\alpha,e} \rightsquigarrow \mathcal{H}^{IM}$. Assume that we have chosen such a collection of STMs.

The extended affine Hecke algebras $\mathcal{H}^{u,\mathfrak{s}_\alpha,e}$ of positive rank which appear as summand of $\mathcal{H}_{\mathrm{uni}}(G)$ are: $E_6[q]$ (for $\mathfrak{s}^1_\emptyset$), $A_2[q^4]$ (for $\mathfrak{s}^1_{D_4}$), $G_2(1,3)[q]$ (for $\mathfrak{s}^u_\emptyset$ with $u \neq 1$). It turns out that for each $u \in \Omega$, G^u has 21 unipotent discrete series representations.

Table 1 displays for each $X^*_{\mathrm{un}}(G) = \Omega^*$-orbit of discrete unramified Langlands parameters: A representative λ, its isotropy group Ω^*_λ, the group A_λ, and for each $u \in \Omega$, the STMs $\Phi^{u,\alpha}_{\mathfrak{s}}$ which contribute to the corresponding packet Π^u_λ of unipotent discrete series of G^u. The argument of the STM indicates the corresponding central character of $\mathcal{H}^{u,\mathfrak{s}_\alpha,e}$, expressed in terms of central characters of graded Hecke algebras via [56, Theorem 8.7], using standard notations referring to distinguished nilpotent orbits for equal parameter cases, and notations for a corresponding generic linear central character as in [56, Section 6] otherwise.

We choose the packets $\Pi^u_\lambda := \Pi^u_{W_0 r_\lambda, \mathbf{L}}$ compatibly with respect to the $X^*_{\mathrm{un}}(G)$-action, but the precise composition of the Π^u_λ depends on the choices of the STMs $\Phi^{u,\alpha}_{\mathfrak{s}}$. Recall from Sect. 2.3 that their parameterization by the elements of $\mathrm{Irr}^u(A_\lambda)$ is chosen in a $X^*_{\mathrm{un}}(G)$-invariant way. By this requirement it suffices to fix the parameterization of the Π^u_λ for a set of representatives λ of the $X^*_{\mathrm{un}}(G)$-orbits of discrete unramified Langlands parameters. With the choices made above, the parameterization of the packets Π^1_λ is determined if we also agree that the generic member of Π_λ corresponds to the trivial representation of A_λ. For $u \neq 1$ and $\lambda = A_1A_5$ or A_2^3, more information is needed to determine the exact parameterization of the sets Π^u_λ (of size 2 and 3 respectively) by a local system as in [40]. Since A_λ is abelian here, Theorem 4.11 is independent of such choices. Therefore, we ignore this issue here.

4.4.2 The parameterization for inner forms of the split adjoint group G of type E_7

We use the same setup and notations as for the case of E_6. The inner forms of G are parameterized by $u \in \Omega \approx C_2$. We have $X^*_{\mathrm{un}}(G) = \Omega^*$. For $u = 1$ we have the following $X^*_{\mathrm{un}}(G)$ orbits of unipotent types: $\mathfrak{s}^1_{\emptyset}, \mathfrak{s}^1_{\mathrm{D}_4}, \mathfrak{s}^1_{\mathrm{E}_6[\theta]}, \mathfrak{s}^1_{\mathrm{E}_6[\theta^2]}, \mathfrak{s}^1_{\mathrm{E}_7[\xi]}, \mathfrak{s}^1_{\mathrm{E}_7[-\xi]}$. For $u = -1$ we have the following orbits of unipotent types: $\mathfrak{s}^u_{\emptyset}, \mathfrak{s}^u_{{}^2\mathrm{A}_5}, \mathfrak{s}^u_{{}^2\mathrm{E}_6[1]}, \mathfrak{s}^u_{{}^2\mathrm{E}_6[\theta]}, \mathfrak{s}^u_{{}^2\mathrm{E}_6[\theta^2]}$. The orbit of $\mathfrak{s}$ is a torsor for $(\Omega^{\mathfrak{s}}_1)^*$ (a quotient of $X^*_{\mathrm{un}}(G)$). By inspection we check that the analog of Remark 4.9 again holds.

The extended affine Hecke algebras $\mathcal{H}^{u,\mathfrak{s}_\alpha,e}$ of positive rank which appear a as summand of $\mathcal{H}_{\mathrm{uni}}(G)$ are for $u = 1$: $\mathrm{E}_7[q]$ (for $\mathfrak{s}^1_{\emptyset}$), $\mathrm{B}_3(4,1)[q]$ (for $\mathfrak{s}^1_{\mathrm{D}_4}$), $\mathrm{C}_1(9,9)[q]$ (for $\mathfrak{s}^1_{\mathrm{E}_6[\theta^i]}$), and moreover for $u = -1$: $\mathrm{F}_4(1,2)[q]$ (for $\mathfrak{s}^{-1}_{\emptyset}$), and $\mathrm{C}_1(9,7)[q]$ (for $\mathfrak{s}^{-1}_{{}^2\mathrm{A}_5}$).

For each $u \in \Omega$, G^u has 44 unipotent discrete series representations. See Table 2. In order to understand the $u = -1$ cases of $\lambda = \mathrm{A}_1\mathrm{D}_6, \mathrm{A}_1\mathrm{D}_6[93], \mathrm{A}_1\mathrm{D}_6[75]$, the following remark is important:

Remark 4.10 The STMs $\Phi^{-1,\pm 1}_{\emptyset} : \mathrm{F}_4(1,2)[q] \rightsquigarrow \mathrm{E}_7[q]$ were constructed at the end of paragraph 3.2.4. Let us write $\Phi := \Phi^{-1,\pm 1}_{\emptyset}$, and let Ψ denote the nontrivial essentially strict spectral automorphism of $\mathrm{E}_7[q]$. Then Φ has the following remarkable property (which is easy to check knowing the spectral map diagram): Let $\lambda_{[3]}, \lambda_{[111]}, \lambda_{[21]}$ be the three orbits of residual points of type $\mathrm{A}_1 \times \mathrm{C}_3$, and let $\mu_{[4]}, \mu_{[31]}, \mu_{[22]}$ be the three orbits of residual points of type B_4. Enumerate these as λ_i and μ_i $(i = 1, 2, 3)$ in this order. Then $\Phi_Z(\lambda_i) = (\Psi_Z \circ \Phi_Z)(\mu_i)$ for all i.

The precise constituents of the packets Π^u_λ depend on the choices of the STMs $\Phi^{u,\alpha}_{\mathfrak{s}}$. Again the exact parameterization of the packets by $\mathrm{Irr}^u(A_\lambda)$ is not uniquely determined for all λ and u. If A_λ is abelian, this does not affect the statement of Theorem 4.11, and we ignore this problem here (but: see [13]). But for $\lambda = \mathrm{E}_7(a_5)$ and $u = -1$ we need to be more careful. This packet corresponds to the generic central character f_8 (notation as in [56, Section 7]) of $\mathrm{F}_4(1,2)[q]$. As was explained in [60], [56, Section 7], [13, paragraph 3.5.2], there are *two* algebraic generic parameter families δ'_8 and δ''_8 of irreducible discrete series characters of $\mathrm{F}_4(m_1, m_2)[q]$ which stay as irreducible discrete series for all $m_1, m_2 > 0$ (and in particular the corresponding $W_0(\mathrm{F}_4)$-types are independent of the parameters). One of these (δ'_8 say) is 10-dimensional, and specializes at equal parameters for F_4 to the discrete series [60] with Langlands parameters $(\mathrm{F}_4(a3), [4])$. The other, δ''_8 restricts to the discrete series with Langlands parameters $(\mathrm{F}_4(a3), [22])$. Comparing with the tables in [64], we see that δ'_8 corresponds with $(\mathrm{E}_7(a_5), -[3])$, while δ''_8 corresponds to $(\mathrm{E}_7(a_5), -[21])$. On the other hand, by [56] and [13] we conclude that $\mathrm{fdeg}(\delta''_8) = 2\mathrm{fdeg}(\delta'_8)$, and this is also equal to $2\mathrm{fdeg}({}^2\mathrm{E}_6[1])$. In view of the above Langlands parameters, this is in accordance with the conjecture [26, Conjecture 1.4].

4.5 Parameterization for non-split quasisplit exceptional groups

The parameterization and the STMs for the remaining twisted exceptional cases were discussed in 3.2.5. By Corollary 4.4, Corollary 4.5 and Remark 3.7 it then follows that Lusztig's parameterization of $\Pi_{W_0 r_{\mathrm{L}}}$ is uniquely determined by this, so this gives rise to a canonical matching of Lusztig's parameterization and our parameterization.

Table 2 The packets Π_λ^u for type E_7 and the contributing STMs

λ	Ω_λ^*	A_λ	STMs for Π_λ^u
E_7	1	C_2	$u=1:\Phi_{\emptyset}^{1,1}(E_7)$ $u=-1:\Phi_{\emptyset}^{u,1}(f_1)$
$E_7(a_1)$	1	C_2	$u=1:\Phi_{\emptyset}^{1,1}(E_7(a_1))$ $u=-1:\Phi_{\emptyset}^{u,1}(f_2)$
$E_7(a_2)$	1	C_2	$u=1:\Phi_{\emptyset}^{1,1}(E_7(a_2))$ $u=-1:\Phi_{\emptyset}^{u,1}(f_3)$
$E_7(a_3)$	1	$S_2\times C_2$	$u=1:\Phi_{\emptyset}^{1,1}(E_7(a_3))$ $u=-1:\Phi_{\emptyset}^{u,1}(f_4)$
$E_7(a_4)$	1	$S_2\times C_2$	$u=1:\Phi_{\emptyset}^{1,1}(E_7(a_4))$ $u=-1:\Phi_{\emptyset}^{u,1}(f_6)$
$E_7(a_5)$	1	$S_3\times C_2$	$u=1:\Phi_{\emptyset}^{1,1}(E_7(a_5))$ $u=-1:\Phi_{\emptyset}^{u,1}(f_8);\ \Phi_{{}^2E_6[1]}^{u,1}$
A_1D_6	1	$C_2\times C_2$	$u=1:\Phi_{\emptyset}^{1,1}(A_1D_6);\ \Phi_{D_4}^{1,1}(B_3)$ $u=-1:\Phi_{\emptyset}^{u,1}(A_1C_3);\ \Phi_{\emptyset}^{u,-1}(B_4)$
$A_1D_6[93]$	1	$C_2\times C_2$	$u=1:\Phi_{\emptyset}^{1,1}(A_1D_6[93]);\ \Phi_{D_4}^{1,1}(B_3[111])$ $u=-1:\Phi_{\emptyset}^{u,1}(A_1C_3[111]);\ \Phi_{\emptyset}^{u,-1}(B_4[31])$
$A_1D_6[75]$	1	$C_2\times C_2$	$u=1:\Phi_{\emptyset}^{1,1}(A_1D_6[75]);\ \Phi_{D_4}^{1,1}(B_3[21])$ $u=-1:\Phi_{\emptyset}^{u,1}(A_1C_3[21]);\ \Phi_{\emptyset}^{u,-1}(B_4[22])$
A_2A_5	1	$C_3\times C_2$	$u=1:\Phi_{\emptyset}^{1,1}(A_2A_5);\ \Phi_{E_6[\theta^i]}^{1,1}$ $u=-1:\Phi_{\emptyset}^{u,1}(A_2A_2);\ \Phi_{{}^2E_6[\theta^i]}^{u,1}$
$A_3^2A_1$	C_2	$C_4\times C_2$	$u=1:\Phi_{\emptyset}^{1,1}(A_3^2A_1);\ \Phi_{D_4}^{1,1}(A_1^3);\ \Phi_{E_7[\pm\xi]}^{1,1}$ $u=-1:\Phi_{\emptyset}^{u,\pm1}(A_3A_1);\ \Phi_{{}^2A_5}^{u,\pm1}(A_1)$
A_7	C_2	C_4	$u=1:\Phi_{\emptyset}^{1,1}(A_3^2A_1);\ \Phi_{D_4}^{1,1}(A_3)$ $u=-1:\Phi_{{}^2A_5}^{u,\pm1}(A_1')$

4.6 Formal degree of unipotent discrete series representations

The application in this section is independent of the uniqueness result based on [19].

A general conjecture has been put forward by [26] expressing the formal degree of a discrete series character in terms of the adjoint gamma factor (also see [20]). Recall that our standing assumption is that **G** is a connected, absolutely simple algebraic group of adjoint type, defined and quasisplit over k, and split over K.

In order to formulate the conjecture in our setting, we first should note that the Haar measures in [26] are equal to those we have used (following [17]) times $v^{-\dim(\mathbf{G})}$. Hence the formal degrees in [26] are $v^{\dim(\mathbf{G})}$ times the formal degree in our setting. Let G^u be an inner form of G. Given a discrete unramified local Langlands parameter λ for G, we defined $\mathcal{A}_\lambda$ (see 2.3). Suppose that for an irreducible representation $\rho \in \mathrm{Irr}(\mathcal{A}_\lambda^u)$ we have a corresponding unipotent (or unramified) discrete series representation $\pi_{(\lambda,\rho)}$ of G^{F_u}, satisfying the expected character identities as asserted in the local Langlands conjecture.

Then [26, Conjecture 1.4] (also see [20, Conjecture 7.1]) is equivalent to (with our normalization of Haar measures):

$$\mathrm{fdeg}(\pi_{(\lambda,\rho)}) = \pm\frac{\dim(\rho)}{|A_\lambda|}v^{-\dim(\mathbf{G})}\gamma(\lambda) \tag{37}$$

where γ denotes the adjoint gamma factor of the discrete local Langlands parameter λ. Following [26, Lemma 3.4], it is easy to show that (using the notations of 2.3)

$$\gamma(\lambda) = \pm v^{\dim(\mathbf{G})}(\mu^{IM})^{(\{r\})} \tag{38}$$

where we should remind the reader that the normalization of the μ-function μ^{IM} of $\mathcal{H}^{IM}(G)$ is given by the trace τ^{IM} such that $\tau^{IM}(1) = \mathrm{Vol}(\mathbb{B}^F)^{-1}$. It was verified in [26] that Reeder's results [60] for Iwahori spherical discrete series representations of adjoint, split exceptional groups over a non-archimedean field are compatible with the conjecture. We are now able to extend this result to arbitrary adjoint absolutely simple groups over a non-archimedean local field which split over an unramified field extension.

Theorem 4.11 *Conjecture* [26, Conjecture 1.4] *[equivalent to equation (37)] holds for all unipotent discrete series representations of inner forms G^u of an unramified connected absolutely simple group G of adjoint, type defined over a non-archimedean local field k, where we use Lusztig's parametrization of unipotent discrete series representations as Langlands parameters.*

Proof We need to consider the classical groups, the nontrivial inner forms of split exceptional groups, and the non-split quasisplit exceptional groups. The way in which we assign unramified discrete Langlands parameters to the members of the packets $\Pi_{W_0 r_{\lambda,\mathbf{L}}}$ of discrete series characters of Definition 4.3 for these cases was explained in Sects. 4.3, 4.4 and 4.5.

We know that $\pi_{(\lambda,\rho)}$ corresponds via Lusztig's arithmetic-geometric correspondences to an irreducible discrete series representation $\delta_{\lambda,\rho}$ of an extended affine Hecke algebra of type $\mathcal{H}^{u,\mathfrak{s},e}$ for some cuspidal type $\mathfrak{s}$ of G^u. By our main Theorem 3.4, there exists an STM $\phi : \mathcal{H}^{u,\mathfrak{s},e} \rightsquigarrow \mathcal{H}^{IM}(G)$ such that $\phi_Z(\mathrm{cc}(\delta_{\lambda,\rho})) = W_0 r$, and we have

$$\mathrm{fdeg}(\pi_{(\lambda,\rho)}) = \mathrm{fdeg}_{\mathcal{H}^{u,\mathfrak{s},e}}(\delta_{(\lambda,\rho)}) = c_{(\lambda,\rho)}(\mu^{IM})^{(\{r\})}. \tag{39}$$

for some rational constant $c_{(\lambda,\rho)} \in \mathbb{Q}$. Combining (37) and (38) we see that what is necessary to verify in order to prove the conjecture in these cases is that

$$c_{(\lambda,\rho)} = \pm\frac{\dim(\rho)}{|A_\lambda|} \tag{40}$$

In [20, Section 5.1] it was shown that

$$\gamma(\lambda) = |C_\lambda^F| q^{N_\lambda}\gamma(\lambda)_q \tag{41}$$

with $\gamma(\lambda)_q$ a q-rational number, $N_\lambda \in \mathbb{N}$ (which is in fact always 0 with our definition of q-rational numbers, but this is not important here), and $C_\lambda^F \subset A_\lambda$ a normal subgroup such that

$$A_\lambda/C_\lambda^F \approx (\pi_0(M_\lambda))^F, \tag{42}$$

is the group of F-fixed points in the component group of the centralizer M_λ of $\lambda|_{\mathrm{SL}_2(\mathbb{C})}$ in $G^\vee$. (The group C_λ^F is the group of F-fixed points in the identity component M_λ^0 of M_λ (a torus).) With this notation we are reduced to proving that

$$\mathrm{fdeg}_{\mathcal{H}^{u,\mathfrak{s},e}}(\delta_{(\lambda,\rho)}) \sim \pm\frac{\dim(\rho)}{|(\pi_0(M_\lambda))^F|}q^{N'_\lambda} \tag{43}$$

(where $\sim$ refers to asymptotic behavior if q tends to 0) for some $N'_\lambda \in \mathbb{N}$. Let us write λ_{ad} for the composition of λ with the canonical homomorphism of $G^\vee$ to $G^\vee_{\mathrm{ad}}$. In the twisted cases it is helpful to note that $A_\lambda/{}^LZ$ is the centralizer of $\lambda_{\mathrm{ad}}|_{\mathrm{SL}_2(\mathbb{C})}$ in the identity component $C_{G^\vee_{\mathrm{ad}}}(\lambda_{\mathrm{ad}}(F))_0$, and realizing that $\lambda_{\mathrm{ad}}(F)$ is a semisimple element of $G^\vee_{\mathrm{ad}} \rtimes \langle\theta\rangle$ of the form (s,θ), where s is a vertex of the alcove of the restricted root system R_0^θ consisting of roots of R_0 restricted to $\mathfrak{t}^\theta$, extended to an affine reflection group by the lattice of translations obtained from projecting the coweight lattice $P(R_0^\vee)$ onto $\mathfrak{t}^\theta$ (see [61]). The semisimple centralizers $C_{G^\vee_{\mathrm{ad}}}(\lambda_{\mathrm{ad}}(F))$ are described by Reeder in [61].

This amounts to a long list of case-by-case verifications. The case of PGL_{n+1} is easy. For u of order $(m+1)|(n+1)$ we have $\Omega_1^{\mathfrak{s},e} = \langle u\rangle \approx C_{m+1}$. Hence $\mathcal{H}^{IM}(G^u)$ is isomorphic to a direct sum of $m+1$ copies of $A_d[q^{m+1}]$ (with $n+1 = (d+1)(m+1)$), normalized by $\tau(1) = (m+1)^{-1}[m+1]_q^{-1}$ (cf. 2.2.3 and Proposition 2.5). This yields $n+1$ unipotent discrete series characters, each with formal degree $(n+1)^{-1}[n+1]_q^{-1}$. In total we thus obtain $n+1$ packets of unipotent discrete series characters, each with $n+1$ members (one element for each inner form).

The case of $G = \mathrm{PU}_{2n}$ or $G = \mathrm{PU}_{2n+1}$ is easy too, since all unipotent affine Hecke algebras are in a generic parameter situation here, in the sense of [11]. It is shown in [11] that the rational constant in the formal degree is then independent of the particular discrete series we consider (of a given Hecke algebra of this type). Looking at the Steinberg character [52, Equation (6.26)] we easily check therefore that the rational constants for all unipotent discrete series are equal to $|{}^LZ|^{-1}$ (so $\frac{1}{2}$ if n is odd, and 1 otherwise).

In the remaining classical cases, one also uses the results of [11], where it was shown that the rational constant factor of the formal degree of a generic discrete series representation of a generic multi parameter type C_n affine Hecke algebra specialized at non-special parameters is equal for all generic discrete series. All rational constants for the discrete series representations of affine Hecke algebras with special parameters can subsequently be computed by this result by a limit procedure, since [56] shows that any discrete serie representation is the limit of a generic continuous family of discrete series representations in a small open set in the parameter space, and that the formal degree is locally continuous in the parameters. We will use Propositions 4.7 and 4.8 to build the packets from the various unipotent Hecke algebras, and compute the expected rational constants according to [26, Proposition 1.4]. On the spectral side, one again relies on the results of [11] and [56] to compute the rational constants.

For exceptional cases, the results of [60] prove the statement for the unipotent discrete series of all split adjoint groups G. For the non-split cases, and the nontrivial inner forms of E_6 and E_7 more work needs to be done, but this follows the same scheme as discussed above, with the help of [61], the tables in Sects. 4.4 and 4.5, and the results of [13,56]. That is, we need to compute the rational constants for the formal degrees of discrete series representations of the multi parameter affine Hecke algebras arising from these non-split cases. The classical Hecke algebras are treated as before, so that leaves the exceptional unequal parameter Hecke algebras which appear in this way.

We find that we need to compute the formal degrees of $\mathrm{G}_2(3,1)$ (for type ${}^3\mathrm{D}_4$), $\mathrm{G}_2(1,3)$ (for type ${}^3\widetilde{\mathrm{E}}_6$), $\mathrm{F}_4(2,1)$ (for type ${}^2\mathrm{E}_6$), and $\mathrm{F}_4(1,2)$ (for type ${}^2\widetilde{\mathrm{E}}_7$). The first main observation in this kind of computations is the fact [56] that any discrete series character δ defines a *generic central character* $gcc(\delta) = W_0r$ (an orbit of generic residual points) and extends uniquely to a continuous family of discrete series characters on a connected component C of the open subset of the space of positive parameters of the Hecke algebra on which W_0r is still residual. Moreover $\mathrm{fdeg}(\delta)$ depends continuously on the parameters in such a continuous family of discrete series.

But there is a deeper fact which is very useful. The formal degree of a generic family of discrete series representations (in the sense of [56]) depends algebraically on the parameters, and this expression only depends on the elliptic class of the limit $\mathbf{q} \to 1$ of the discrete series representation (a representation of W). This result follows essentially from [14] and the Euler–Poincaré formula in [56], using the argument of [12, Proposition 5.6] in the unequal parameter setting. This implies (see [13] for details) that the formal degree of generic families associated with the same generic central character W_0r but defined on different connected components C and C' of the open subset of the positive parameter space where W_0r is residual is given by *the same* algebraic expression (provided the families define the same elliptic representation of W), except possibly for a sign change. (This result generalizes the result of [11] to arbitrary Hecke algebras). This algebraic expression for the formal degree is a product formula (see [56]) of terms $(1 \pm M)^{\pm 1}$ where M is a monomial in the parameters, multiplied by a rational constant d (which only depends on an elliptic representation of W), a monomial in the parameters, and a sign.

The upshot is that in order to compute $\mathrm{fdeg}(\delta)$ it is sufficient to compute $\mathrm{fdeg}(\delta')$ for any discrete series δ' with $gcc(\delta') = W_0r$ at any positive parameter q' where $W_0r(q')$

is residual, provided δ and δ' define the same elliptic representation of W. Using the results of [60], we can find a δ' and q' where the constants are known for every generic family. Hence the generic rational constants d can be determined, and from this we can determine the formal degree at any singular parameter line in the parameter space by continuity.

See Example 4.6.3 for more details in the case ${}^3\mathrm{D}_4$. For $\mathrm{F}_4(2,1)$ we have a similar situation, here we need to cross the singular lines $\frac{k_1}{k_2} = \frac{6}{5}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}$ in the parameter space. For this we need to know the confluence relations of the generic discrete series at these singular lines. This can be deduced from [56, Table 3]. The considerations are similar as in Example 4.6.3. Similarly for ${}^3\widetilde{\mathrm{E}}_6$ and ${}^2\widetilde{\mathrm{E}}_7$. □

4.6.1 Unipotent representations of inner forms of $PCSp_{2n}$, $P(CO^0_{2n})$, $P((CO^*_{2n+2})^0)$
In these cases, a unipotent affine Hecke algebra is always isomorphic to a direct sum of finitely many copies of a normalized affine Hecke algebra which is related to an object of $\mathfrak{C}^{\mathrm{III}\cup\mathrm{V}}_{\mathrm{class}}$ or $\mathfrak{C}^{\mathrm{IV}\cup\mathrm{VI}}_{\mathrm{class}}$ through a (finite) sequence of spectral covering maps.

Let us first compute the rational factors appearing in the formal degrees of discrete series representations of a normalized affine Hecke algebra $\mathcal{H}$ of type $\mathrm{C}_d(m_-, m_+)[q]$ with $m_\pm \in \mathbb{Z}$, normalized by $\tau(1) = 1$. Using the group $\mathbb{D}_8$ of spectral isomorphisms (see [54, Remark 7.7]) we may, without loss of generality, assume that $0 \le m_- \le m_+$.

As described in Sect. 4.2, the discrete series of $\mathcal{H}$ are parameterized by ordered pairs (σ_-, σ_+) of symbols associated to an ordered pair (u_-, u_+) of distinguished unipotent partitions for the pair of parameters $m = (m_-, m_+)$ (so $u_\pm$ is a partition of $m_\pm^2 + 2d_\pm$). By Slooten's "joining procedure" [63, Theorem 5.27] (see also the explanation in Sect. 4.2), the set of symbols $\sigma_\pm$ corresponds bijectively to the set of partitions $\pi_\pm$ of $d_\pm$ whose $m_\pm$-tableaux have distinct extremities in the sense of [63] and such that the corresponding orbit of linear residual points corresponds to $u_\pm$. Then the vector consisting of the contents of the boxes of this $m_\pm^\epsilon := m_\pm + \epsilon_\pm$-tableau of $\pi_\pm$ defines, for all $\epsilon_\pm$ sufficiently small, a linear residual point $\xi_\pm(m_\pm + \epsilon_\pm)$ whose $W_{n_\pm}$-orbit generically supports a unique discrete series character. We will denote the discrete series character by $\delta_{(\pi_-,\pi_+)}(\epsilon_-, \epsilon_+)$.

Theorem 4.12 *Let $m = (m_-, m_+) \in \mathbb{Z}^2$ be such that $0 \le m_- \le m_+$. Consider $\pi_{(u_-,u_+),(\sigma_-,\sigma_+)} := \delta_{(\pi_-,\pi_+)}(0,0)$ as a discrete series of the normalized affine Hecke algebra $(\mathcal{H}, \tau)$ of type $C_d(m_-, m_+)[q]$, normalized by $\tau(1) = 1$. Let (u_-, u_+) be the pair of unipotent partitions of type $m = (m_-, m_+)$ associated with the pair $(T_{m_-}(\pi_-), T_{m_+}(\pi_+))$ of m-tableaux, and let $(\xi_-(m_- + \epsilon_-), \xi_+(m_+ + \epsilon_+))$ be the corresponding pair of linear residual points. Let $fdeg_{\mathbb{Q}}(\pi_{(u_-,u_+),(\sigma_-,\sigma_+)})$ denote the rational factor of $fdeg(\pi_{(u_-,u_+),(\sigma_-,\sigma_+)})$. Let $u_-\cup u_+$ be the partition which one obtains by concatenating u_- and u_+ and rearranging the parts as a partition (our convention will be to arrange the parts in a nondecreasing order). Let $\#(u)$ denote the number of distinct parts of a partition u. Then we have*

$$fdeg_{\mathbb{Q}}(\pi_{(u_-,u_+),(\sigma_-,\sigma_+)}) = 2^{-\#(u_-\cup u_+)+m_+}. \tag{44}$$

Proof Let the central character of $\delta_{(\pi_-,\pi_+)}(\epsilon_-, \epsilon_+)$ be denoted by $W_0 r$, where $r := r_{(\pi_-,\pi_+)}(\epsilon_-, \epsilon_+) = (-r_-(\epsilon_-), r_+(\epsilon_+))$ with $r_\pm(\epsilon_\pm) := \exp(\xi_\pm(m_\pm + \epsilon_\pm))$. We have

$\mathrm{fdeg}(\delta_{(\pi_-,\pi_+)}(\epsilon_-,\epsilon_+)) = cm_{W_0r}$ by [56, Theorem 4.6], with $c \in \mathbb{Q}^\times$, and with the rational function m_{W_0r}, defined by [56, (39)]. The constant $|c|$ is known [11, Theorem C] and turns out to be equal to 1, independent of the parameters and of (π_-,π_-) (there is a harmless but unfortunate mistake in [11, Definition 4.3] (the factor $\frac{1}{2}$ on the right-hand side should not be there, see the update on arXiv) which resulted in the erroneous extra factor $\frac{1}{2}$ in [11, Theorem C]). We have the basic regularity result [56, Corollary 4.4]. Hence $\mathrm{fdeg}(\pi_{(u_-,u_+),(\sigma_-,\sigma_+)})$ equals the limit for $(\epsilon_-,\epsilon_+) \to (0,0)$ of m_{W_0r}.

For an arbitrary root datum $\mathcal{R}$ with parameter function $m^\epsilon_\pm(\alpha) = m_\pm(\alpha) + \epsilon_\pm(\alpha)$, and a generic residual point r which specializes to a residual point at $\epsilon_\pm = 0$, we can rewrite m_{W_0r} in the following form (cf. [54, (13)])) (here $N = N(\epsilon)$ is an affine linear function of the deformation parameters ϵ):

$$m_{W_0r} =$$
$$v^N \prod_{\alpha\in R_{0,+}} \frac{(1+\alpha(r))^2(1-\alpha(r))^2}{(1+q^{m^\epsilon_-(\alpha)}\alpha(r))(1+q^{-m^\epsilon_-(\alpha)}\alpha(r))(1-q^{m^\epsilon_+(\alpha)}\alpha(r))(1-q^{-m^\epsilon_+(\alpha)}\alpha(r))}$$

where a factor of the numerator or of the denominator has to be omitted if it is identically equal to 0 as a function of ϵ in a neighborhood of 0.

In our present case, $R_{0,+} = \{e_i \pm e_j \mid 1 \le i < j \le d\} \cup \{e_i \mid 1 \le i \le d\}$.

For a positive root α of type D, we have $m^\epsilon_-(\alpha) = 0$ and $m^\epsilon_+(\alpha) = 1$; for positive root β of type A_1^d, we have $m^\epsilon_-(\beta) = m_- + \epsilon_-$ and $m^\epsilon_+(\beta) = m_+ + \epsilon_+$. In the limit $\epsilon = (\epsilon_-,\epsilon_+) \to 0$, some of the factors which are generically nonzero tend to 0, but the number of those factors in the numerator and denominator is equal by [56, Corollary 4.4] (or [53]). This potentially produces rational factors in the limit, but actually all such factors (for type D roots as well as for type A_1^d roots) are of the form $(1-q^{\pm 2\epsilon_-})$, $(1-q^{\pm 2\epsilon_+})$, or $(1-q^{\pm(\epsilon_- - \epsilon_+)})$. For each of these three types, the total number of these factors in the numerator and denominator has to be equal by the above regularity result. Hence altogether these factors yield at most a sign in the limit, and that does not contribute to $\mathrm{fdeg}_{\mathbb{Q}}(\pi_{(u_-,u_+),(\sigma_-,\sigma_+)})$. In addition we have factors $(1+q^{l(\epsilon)})$, with $l(\epsilon)$ linear in ϵ, in the denominator and numerator. Each such factor yields a factor 2, regardless of the precise form of $l(\epsilon)$. Let the total number of factors 2 thus obtained be denoted by M. In order to count M, let us write $h^{m_\pm}_{u_\pm}(x)$ for the number coordinates of $\xi_\pm(m_\pm)$ which are equal to x (for $x \in \mathbb{Z}_{\ge 0}$) (cf. [22], or [56, Proposition 6.6]). We also define $H^{m_\pm}_{u_\pm}(x) = h^{m_\pm}_{u_\pm}(x)$ for $x > 0$, and $H^{m_\pm}_{u_\pm}(0) = 2h^{m_\pm}_{u_\pm}(0)$. Finally, if h is a function on $\mathbb{Z}$, we define $\delta(h)(x) := h(x) - h(x+1)$. It is straightforward to deduce from the above formula for m_{W_0r} that

$$\begin{aligned}
M &:= \sum_{x\ge 0} \delta(H^{m_-}_{u_-})(x)\delta(H^{m_+}_{u_+})(x) - H^{m_-}_{u_-}(m_+) - H^{m_+}_{u_+}(m_-) \\
&= \sum_{x\ge 1} \delta(H^{m_-}_{u_-})(x)\delta(H^{m_+}_{u_+})(x) + \delta(H^{m_-}_{u_-})(0)\delta(H^{m_+}_{u_+})(0) - H^{m_-}_{u_-}(m_+) - H^{m_+}_{u_+}(m_-) \\
&= \sum_{x\ge 1} \delta(H^{m_-}_{u_-})(x)\delta(H^{m_+}_{u_+})(x) + (J^{m_-}_{u_-}(0) + \delta_{m_-,0} - 1)(J^{m_+}_{u_+}(0) + \delta_{m_+,0} - 1)
\end{aligned}$$

$$- H^{m_-}_{u_-}(m_+) - H^{m_+}_{u_+}(m_-)$$

where $J^m_u(0) = 1$ if 1 is a part of u (equivalently, if 0 is a jump of ξ), and $J^m_u(0) = 0$ otherwise (this value depends only on u (is independent of m)). Recall that ([63], or [56, Proposition 6.6]) the number of jumps of the vector of contents $\xi(m)$ of $T_m(\pi)$ equals $\#(u)$, and that this is also equal to $m + H^m_u(0)$. In the second equality above we used that $\delta(H^m_u)(0) = 2h^m_u(0) - h^m_u(1) = J^m_u(0) + \delta_{m,0} - 1$.

Now let $\delta_\pm \in \{0, 1\}$ be such that $\delta_\pm \equiv m_\pm \mathrm{mod}(2)$. There exist partitions $\pi'_\pm$ such that set of jumps of the vector $\xi'_\pm$ of contents of the $\delta_\pm$-tableau $T_{\delta_\pm}(\pi'_\pm)$ of $\pi'_\pm$ equals the set of jumps of $\xi_\pm$ (cf. [56, Proposition 6.6]). By Proposition 4.7, the central character $W'_0 r'$ of $\mathrm{C}_n(\delta_-, \delta_+)[q]$ (with $2n = |u_-| + |u_+| - \delta_- - \delta_+$) which corresponds to $W_0 r$ under the translation STM $\mathrm{C}_d(m_-, m_+)[q] \rightsquigarrow \mathrm{C}_n(\delta_-, \delta_+)[q]$, is of the form $r' = (-\exp(\xi'_-), \exp(\xi'_+))$. Let $h^{\delta_\pm}_{u_\pm}(x)$ denote the multiplicity of x in the vector $\xi'_\pm$, and let $H^{\delta_\pm}_{u_\pm}(x)$ be defined, similar to $H^{m_\pm}_{u_\pm}(x)$. We define $\Delta^{m_\pm}_\pm := H^{\delta_\pm}_{u_\pm}(x) - H^{m_\pm}_{u_\pm}(x)$. Then it follows from the definition of the jump vector at $m_\pm$ and at $\delta_\pm$ that for $x \geq 1$, $\Delta^{m_\pm}_\pm(x) = \max(0, m - x)$. Thus for $x \geq 1$, we have $\delta(\Delta^{m_\pm}_\pm)(x) = \chi_{[1,m_\pm-1]}(x)$, where $\chi_{[1,m_\pm-1]}$ denotes the indicator function of the interval $[1, m_\pm - 1]$. Let $\#(u_- \cap u_+)$ denote the number of parts that u_- and u_+ have in common. Then we get

$$\begin{aligned}
M &:= \sum_{x\geq 1} \delta(H^{\delta_-}_{u_-})(x)\delta(H^{\delta_+}_{u_+})(x) - H^{\delta_-}_{u_-}(1) + H^{\delta_-}_{u_-}(m_+) - H^{m_-}_{u_-}(m_+) \\
&\quad - H^{\delta_+}_{u_+}(1) + H^{\delta_+}_{u_+}(m_-) - H^{m_+}_{u_+}(m_-) + (J^{\delta_-}_{u_-}(0) + \delta_{m_-,0} - 1)(J^{\delta_+}_{u_+}(0) + \delta_{m_+,0} - 1) \\
&\quad\quad - \delta_{m_+,0}(J^{\delta_-}_{u_-}(0) - \delta_-) - \delta_{m_-,0}(J^{\delta_+}_{u_+}(0) - \delta_+) + \max(0, m_- - 1) \\
&= \#(u_- \cap u_+) - H^{\delta_-}_{u_-}(1) - H^{\delta_+}_{u_+}(1) + \Delta^{m_-}_-(m_+) + \Delta^{m_+}_-(m_-) - J^{\delta_-}_{u_-}(0) - J^{\delta_+}_{u_+}(0) \\
&\quad\quad + \delta_{m_-,0}\delta_{m_+,0} + \delta_{m_-,0}(\delta_+ - 1) + \delta_{m_+,0}(\delta_- - 1) + 1 + \max(0, m_- - 1) \\
&= \#(u_- \cap u_+) - H^{\delta_-}_{u_-}(0) - H^{\delta_+}_{u_+}(0) - \delta_- - \delta_+ + \Delta^{m_+}_-(m_-) \\
&\quad\quad + \delta_{m_-,0}\delta_{m_+,0} + \delta_{m_-,0}(\delta_+ - 1) + \delta_{m_+,0}(\delta_- - 1) + 1 + \max(0, m_- - 1) \\
&= -\#(u_- \cup u_+) + \delta_{m_-,0}\Delta^{m_+}_+(0) + (1 - \delta_{m_-,0}))\Delta^{m_+}_+(m_-) \\
&\quad\quad + \delta_{m_-,0}\delta_{m_+,0} + \delta_{m_-,0}(\delta_+ - 1) + \delta_{m_+,0}(\delta_- - 1) + 1 + \max(0, m_- - 1) \\
&= -\#(u_- \cup u_+) + \delta_{m_-,0}(m_+ - \delta_+) + (1 - \delta_{m_-,0}))(m_+ - m_-) \\
&\quad\quad + \delta_{m_-,0}\delta_{m_+,0} + \delta_{m_-,0}(\delta_+ - 1) + \delta_{m_+,0}(\delta_- - 1) + 1 + \max(0, m_- - 1) \\
&= -\#(u_- \cup u_+) + m_+ + \delta_{m_-,0}\delta_{m_+,0} + \delta_{m_+,0}(\delta_- - 1) \\
&= -\#(u_- \cup u_+) + m_+
\end{aligned}$$

finishing the proof. In the above computation we used at several steps that $0 \leq m_- \leq m_+$, and that $H^m_u(0) = \#(u) - m$. □

A similar but easier computation shows a similar result for Hecke algebras of unipotent representations of SO_{2n+1} (cf. 4.6.2).

Theorem 4.13 *Let* $m = (m_-, m_+) \in (\frac{1}{2} + \mathbb{Z})^2$ *be such that* $0 < m_- \leq m_+$. *Consider* $\pi_{(u_-,u_+),(\sigma_-,\sigma_+)} := \delta_{(\pi_-,\pi_+)}(0, 0)$ *as a discrete series of the normalized affine Hecke* *algebra* $(\mathcal{H}, \tau)$ *of type* $C_d(m_-, m_+)[q]$*, normalized by* $\tau(1) = 1$*. Let* (u_-, u_+) *be the pair of distinguished unipotent partitions of type* $m = (m_-, m_+)$ *associated with pair* $(T_{m_-}(\pi_-), T_{m_+}(\pi_+))$ *of m-tableaux (i.e.,* $u_\pm$ *is a partition of* $2n_\pm$ *with distinct, even parts of length at least* $m_\pm - \frac{1}{2}$*, such that* $n_- + n_+ = n$*). Then*

$$fdeg_{\mathbb{Q}}(\pi_{(u_-,u_+),(\sigma_-,\sigma_+)}) = 2^{m_+-\frac{1}{2}-\#(u_-\cup u_+)}. \tag{45}$$

The proof of the next result (of [19]) is similar in spirit as the above results.

Theorem 4.14 [19] *Let* $d = d_- + d_+ \in \mathbb{Z}_{\geq 0}$*, and let* $\pi_\pm \vdash d_\pm$*. Let* $0 \leq m_- \leq m_+$ *with* $m_\pm \in \pm\frac{1}{4} + \mathbb{Z}$*. Let* $m_\pm = \kappa_\pm + \frac{1}{4}(2\epsilon_\pm - 1)$ *with* $\kappa_\pm \in \mathbb{Z}_{\geq 0}$ *and* $\epsilon_\pm \in \{0, 1\}$*. Let* $\delta_\pm \in \{0, 1\}$ *be defined by* $\delta_\pm \equiv \kappa_\pm$ *(mod 2). Consider* $\pi_{(\pi_-,\pi_+),extra} := \delta_{(\pi_-,\pi_+)}(0, 0)$ *as a discrete series of the normalized affine Hecke algebra* $(\mathcal{H}, \tau)$ *of type* $C_d(m_-, m_+)[q^2]$*, normalized by* $\tau(1) = 1$*. Let* (u_-, u_+) *be the pair of unipotent partitions of type* (δ_-, δ_+) *associated with the pair* $(T_{m_-}(\pi_-), T_{m_+}(\pi_+))$ *of m-tableaux via the extraspecial STM [cf. (35), and* [19]*]* $\mathcal{H} \rightsquigarrow C_n(\delta_-, \delta_+)[q]$*. Then we have*

$$fdeg_{\mathbb{Q}}(\pi_{(\pi_-,\pi_+),extra}) = \begin{cases} 2^{\#(u_-\cap u_+)-h_-(\frac{1}{4})-h_+(\frac{1}{4})} & \text{if } \epsilon_- \neq \epsilon_+ \\ 2^{\#(u_-\cap u_+)-h_-(\frac{1}{4})-h_+(\frac{1}{4})-\kappa_-} & \text{if } \epsilon_- = \epsilon_+. \end{cases} \tag{46}$$

Let us now look at the Proof of Theorem 4.11 for these cases:

Lemma 4.15 *Theorem 4.11 holds for* $G = PCSp_{2n}$ *(with* $n \geq 2$*),* $P(CO^0_{2n})$ *(with* $n \geq 4$*) or* $P((CO^*_{2n})^0)$ *(with* $n \geq 4$*).*

Proof Assume that we have fixed a Borel subgroup $B \subset G$, a maximal torus $T \subset B$ and a pinning for the reductive groups G considered below.

For $G = \mathrm{PCSp}_{2n}$, we have $\Omega = \{\epsilon, \eta\} \approx C_2$, hence we need to consider two inner forms G^ϵ and G^η. We first deal with the split form G^ϵ. We have $\mathcal{H}^{IM}(G^\epsilon)$ of type $\mathrm{B}_n(1, 1)[q]$ (also denoted by $\mathcal{H}(\mathcal{R}^{\mathrm{B}}_{\mathrm{ad}}, m^B)$ in [54, 7.1.4]). The conjugacy classes of parahoric subgroups of G^ϵ which carry a (unique) cuspidal unipotent representation correspond to unordered pairs (a, b) with $a, b \in \mathbb{Z}_{\geq 0}$ such that $d := n - a^2 - b^2 - a - b \geq 0$. The corresponding type $\mathfrak{s}_{d,a,b}$ corresponds to a subdiagram of type $\mathrm{B}_{a^2+a} \sqcup \mathrm{B}_{b^2+b}$ of the affine diagram $\mathrm{C}^{(1)}_n$ of a set of affine simple roots of $G^\epsilon(k)$. Consider the corresponding associated normalized (extended) affine Hecke algebra $\mathcal{H}^{\epsilon,\tilde{\mathfrak{s}},e}$. Put $m_- := |a - b|$, and $m_+ := 1 + a + b$. Then

$$\mathcal{H}^{\epsilon,\tilde{\mathfrak{s}},e} \simeq \begin{cases} \mathrm{C}_d(m_-, m_+)[q] & \text{if } a \neq b \\ \mathrm{B}_d(1, m_+)[q] & \text{otherwise} \end{cases}$$

and

$$\Omega^{\mathfrak{s}}_1 \simeq \begin{cases} 1 & \text{if } a \neq b \text{ or } d > 0 \\ C_2 & \text{otherwise.} \end{cases}$$

Thus by Proposition 2.5 and [9, Section 13.7] the rational factor $\tau^{\epsilon,\mathfrak{s},e}(1)_{\mathbb{Q}}$ of the trace $\tau^{\epsilon,\mathfrak{s},e}$ of $\mathcal{H}^{\epsilon,\tilde{\mathfrak{s}},e}$ is such that (since $\mathcal{H}^{\epsilon,\mathfrak{s}} \simeq \mathcal{H}^{\epsilon,\tilde{\mathfrak{s}},e} \otimes \mathbb{C}[\Omega_1^{\mathfrak{s}}]$, cf. Corollary 2.7):

$$\tau^{\epsilon,\mathfrak{s},e}(1)_{\mathbb{Q}} = \begin{cases} 2^{-a-b} & \text{if } a \neq b \text{ or } d > 0 \\ 2^{-1-a-b} & \text{otherwise.} \end{cases}$$

As was discussed in paragraph 3.2.7 (also see [54, 7.1.4]), there exists an STM $\mathcal{H}^{\epsilon,\tilde{\mathfrak{s}},e} \rightsquigarrow \mathcal{H}^{d,a,b}$ corresponding to a strict algebra inclusion $\mathcal{H}^{d,a,b} \subset \mathcal{H}^{\epsilon,\tilde{\mathfrak{s}},e}$, where $\mathcal{H}^{d,a,b} = C_d(m_-, m_+)[q]$ is an object of $\mathfrak{C}^{\mathrm{III}}_{\mathrm{class}}$. This inclusion satisfies

$$\begin{cases} \mathcal{H}^{d,a,b} = \mathcal{H}^{\epsilon,\tilde{\mathfrak{s}},e} & \text{if } a \neq b \text{ or } d = 0 \\ \mathcal{H}^{d,a,b} \subset \mathcal{H}^{\epsilon,\tilde{\mathfrak{s}},e} & \text{has index two, otherwise.} \end{cases}$$

We define the trace $\tau^{d,a,b}$ of $\mathcal{H}^{d,a,b}$ by restriction of the trace $\tau^{\epsilon,\mathfrak{s},e}(1)_{\mathbb{Q}}$ of $\mathcal{H}^{\epsilon,\tilde{\mathfrak{s}},e}$, so we have

$$\tau^{d,a,b}(1)_{\mathbb{Q}} = \begin{cases} 2^{1-m_+} & \text{if } a \neq b \text{ or } d > 0 \\ 2^{-m_+} & \text{otherwise.} \end{cases}$$

Now we want to compute the rational factor of the formal degree of a unipotent discrete series representation π in a block corresponding to the type $\mathfrak{s} := \mathfrak{s}_{d,a,b}$. According to Lusztig's parameterization [40] we attach to π an unramified Langlands parameter λ, and an irreducible representation α of the component group A_λ such that the center ${}^LZ \subset A_\lambda$ acts trivially in this representation (since $u = 1$ here). This is equivalent to α being a one-dimensional representation, and we can parameterize such α by a pair of Lusztig–Shoji symbols (σ_-, σ_+) for a pair (u_-, u_+) of distinguished unipotent partitions for the parameter (m_-, m_+), such that $|u_-| + |u_+| = 2n + 1$. We denote by $\pi^G_{\lambda,(\sigma_-,\sigma_+)}$ the corresponding irreducible discrete series representation of $\mathcal{H}^{\epsilon,\tilde{\mathfrak{s}},e}$ (depending on the chosen isomorphism $\mathcal{H}^{\epsilon,\mathfrak{s}} \simeq \mathcal{H}^{\epsilon,\tilde{\mathfrak{s}},e} \otimes \mathbb{C}[\Omega_1^{\mathfrak{s}}]$). According to [8], the formal degree of π is equal to the formal degree of $\pi^G_{\lambda,(\sigma_-,\sigma_+)}$. As before, let $\mathrm{fdeg}_{\mathbb{Q}}(\pi^G_{\lambda,(\sigma_-,\sigma_+)})$ denote the rational factor of $\mathrm{fdeg}(\pi^G_{\lambda,(\sigma_-,\sigma_+)})$. The irreducible discrete series representations of $\mathcal{H}^{d,a,b}$ with the central character corresponding to (u_-, u_+) are parameterized [56] by pairs of Slooten symbols (σ_-, σ_+) associated to (u_-, u_+) at parameter (m_-, m_+). The discrete series of $\mathcal{H}^{d,a,b}$ corresponding to (u_-, u_+), (σ_-, σ_+) was denoted by $\pi_{(u_-,u_+),(\sigma_-,\sigma_+)}$. By Remark 4.6 we easily check that we have a total of $\binom{l_-+l_+}{(l_-+l_+-1)/2} + \binom{l_-+l_+}{(l_-+l_+-5)/2} + \cdots = 2^{l_-+l_+-2}$ such discrete series representations, in accordance with the number of one-dimensional representations of A_λ (with is of type $2^{(l_-+l_+-3)+2}$, according to Proposition 4.8). By the above, combined with Theorem 4.12 we see that

$$\mathrm{fdeg}_{\mathbb{Q}}(\pi_{(u_-,u_+),(\sigma_-,\sigma_+)}) = \begin{cases} 2^{1-\#(u_-\cup u_+)} & \text{if } a \neq b \text{ or } d > 0 \\ 2^{-\#(u_-\cup u_+)} & \text{otherwise.} \end{cases}$$

According to [56, Paragraph 6.4] (also see [18, Proposition 6.6]), and using the fact that (see [10]) the Slooten symbols and the Lusztig–Shoji symbols match, we see that upon restriction to $\mathcal{H}^{d,a,b}$ there are the following possibilities:

$$\pi^G_{\lambda,(\sigma_-,\sigma_+)}|_{\mathcal{H}^{d,a,b}} = \begin{cases} \pi_{(u_-,u_+),(\sigma_-,\sigma_+)} & \text{if } a \neq b \text{ or } d = 0 \\ \pi_{(u_-,u_+),(\sigma_-,\sigma_+)} & \text{if } a = b,\ d > 0 \text{ and } u_- = 0 \\ \pi_{(u_-,u_+),(\sigma_-,\sigma_+)} \oplus \pi_{(u_-,u_+),(\sigma'_-,\sigma_+)} & \text{if } a = b,\ d > 0 \text{ and } u_- \neq 0. \end{cases}$$

Here σ'_- is the symbol obtained from σ_- by interchanging the top and the bottom rows. In the second case $d > 0$ and $u_- = 0$, there are two irreducible discrete series representations of $\mathcal{H}^{IM}(G^\epsilon)$ which restrict to the same irreducible $\pi_{(u_-,u_+),(\sigma_-,\sigma_+)}$ (whose central characters form one $X^*_{\text{un}}(G^\epsilon)$-orbit). Restriction of the spectral decomposition of $\tau^{\epsilon,\mathfrak{s},e}$ to $\mathcal{H}^{d,a,b}$ shows $\text{fdeg}_{\mathbb{Q}}(\pi^G_{\lambda,(\sigma_-,\sigma_+)}) = \frac{1}{2}\text{fdeg}_{\mathbb{Q}}(\pi_{(u_-,u_+),(\sigma_-,\sigma_+)})$ in this case, while $\text{fdeg}_{\mathbb{Q}}(\pi^G_{\lambda,(\sigma_-,\sigma_+)}) = \text{fdeg}_{\mathbb{Q}}(\pi_{(u_-,u_+),(\sigma_-,\sigma_+)})$ in the other two cases. Hence we have, for all $d \geq 0$,

$$\text{fdeg}_{\mathbb{Q}}(\pi^G_{\lambda,(\sigma_-,\sigma_+)}) = \begin{cases} 2^{-\#(u_-\cup u_+)} & \text{if } u_- = 0 \\ 2^{1-\#(u_-\cup u_+)} & \text{if } u_- \neq 0. \end{cases}$$

Hence, using Proposition 4.8, (42) and (43) we see that Theorem 4.11 follows for this case $G = \text{PCSp}_{2n}$ and $u = \epsilon$, if we show that $|C^{F_\epsilon}_\lambda| = 2^{\#(u_-\cap u_+)}$. Recall that $M^0_\lambda \simeq (\mathbb{C}^\times)^{\#(u_-\cap u_+)}$ (cf. [9, Section 13.1]), on which F_ϵ acts by $\text{Ad}(s_0)$. Clearly $\text{ad}(s_0)$ must act by -1 on $\mathfrak{m}_\lambda = \text{Lie}(M^0_\lambda)$, and so F_ϵ acts by $F_\epsilon(m) = m^{-1}$ on M^0_λ. The desired result follows for $u = \epsilon$.

Next, we need to check Theorem 4.11 for the contributions coming from the nontrivial inner form G^η in this case. Now the cuspidal unipotent parahoric subgroups $\mathbb{P}^\eta_{s,t}$ are given by η-invariant subdiagrams of type $\text{B}_{s^2+s} \cup \text{B}_{s^2+s} \cup A_{\frac{1}{2}(t^2+t)-1}$ such that $d + 1 := \frac{1}{2}(n - 2(s^2+s) - \frac{1}{2}(t^2+t) + 2) \in \mathbb{Z}_{>0}$. This corresponds to a type $\mathfrak{s} := \mathfrak{s}^\eta_{d,s,t}$ for G^η which is completely determined by a pair of nonnegative integers (s,t) satisfying the above inequality. The corresponding affine Hecke algebra $\mathcal{H}^{\eta,\mathfrak{s},e}$ is of type $C_d(m_-,m_+)[q^2]$, with $m_+ = \frac{1}{4}(3 + 2t + 4s)$ and $m_- = \frac{1}{4}|1 - 2t + 4s|$. We have $\Omega^{\mathfrak{s}}_1 = C_2$ (always), and hence using Proposition 2.5 and [9, Section 13.7], the rational factor $\tau^{\eta,\mathfrak{s},e}(1)_{\mathbb{Q}}$ of $\tau^{\eta,\mathfrak{s},e}(1)$ equals

$$\tau^{\eta,\mathfrak{s},e}(1)_{\mathbb{Q}} = 2^{-s-1} = \begin{cases} 2^{-\frac{1}{2}(m_++m_-+1)} & \text{if } \epsilon_- \neq \epsilon_+ \\ 2^{-\frac{1}{2}(m_+-m_-+1)} & \text{if } \epsilon_- = \epsilon_+. \end{cases}$$

Using Theorem 4.14, we obtain two discrete series representations $\pi^{\pm}_{(\pi_-,\pi_+),extra}$, with $\text{fdeg}_{\mathbb{Q}}(\pi^{\pm}_{(\pi_-,\pi_+),extra}) = 2^{\frac{1}{2}(l_-+l_+-1)-\#(u_-\cup u_+)}$ (in all cases). In view of Proposition 4.8, this is indeed the rational factor of the formal degree of the two elements of the Lusztig packet attached to the Langlands parameter λ on which ${}^LZ \subset A_\lambda$ acts by η times the identity, as predicted by (43).

For $G = \mathrm{P}(\mathrm{CO}^0_{2n})$ (with $n \geq 4$) we do a similar analysis. In this case, Ω is isomorphic to C_4 if n is odd, and isomorphic to $C_2 \times C_2$ if n is even. Let θ denote a diagram automorphism of order two of the finite type D_n subdiagram. Let us write $\Omega = \{\epsilon, \eta, \rho, \eta\rho\}$, where η is θ-invariant, and $[\rho, \theta] = [\rho\eta, \theta] = \eta$. Let us first consider the split case G^ϵ. In this case $\mathcal{H}^{IM}(G)$ is of type $\mathrm{D}_n[q]$, which was denoted by $\mathcal{H}(\mathcal{R}^{\mathrm{D}}_{\mathrm{ad}}, m^D)$ in [54, (54)]. Let us denote $\mathcal{H}(\mathcal{R}^{\mathrm{D}}_{\mathbb{Z}^n}, m^D)$ (notation as in [54, Paragraph 7.1.4]) by $\tilde{\mathrm{D}}_n[q]$. Its spectral diagram consists of the Dynkin diagram for $\mathrm{D}^{(1)}_n$, with the action of the automorphism η as in [18, Figure 1] (we have, in the sense of [54, Definition 2.11], that $\Omega^\vee_Y = \langle\eta\rangle \simeq C_2$). As was discussed in [54, Paragraph 7.1.4], we have spectral coverings $\mathrm{D}_n[q] \rightsquigarrow \tilde{\mathrm{D}}_n[q]$ and $\tilde{\mathrm{D}}_n[q] \rightsquigarrow \mathrm{C}_n(0,0)[q]$, corresponding to strict algebra embeddings $\tilde{\mathrm{D}}_n[q] \subset \mathrm{D}_n[q]$ and $\tilde{\mathrm{D}}_n[q] \subset \mathrm{C}_n(0,0)[q]$, both of index 2. We normalize the trace of $\tilde{\mathrm{D}}_n[q]$ by restriction from $\mathrm{D}_n[q]$, and of $\mathrm{C}_n(0,0)[q]$ such that its restriction to $\tilde{\mathrm{D}}_n[q]$ equals the trace we just defined on $\tilde{\mathrm{D}}_n[q]$. The conjugacy classes of parahoric subgroups of G which support a (unique) cuspidal unipotent representation correspond to unordered pairs (a, b) with $a, b \in 2\mathbb{Z}_{\geq 0}$ such that $d = n - a^2 - b^2 \geq 0$. The pair (a, b) corresponds to a subdiagram of type $\mathrm{D}_{a^2} \sqcup \mathrm{D}_{b^2}$ of the type $\mathrm{D}^{(1)}_n$ diagram of a set of simple affine roots of $G(k)$. We put $m_- = |a - b|$, and $m_+ = |a + b|$. We have

$$\mathcal{H}^{\epsilon,\tilde{\mathfrak{s}},e} \simeq \begin{cases} \mathrm{C}_d(m_-, m_+)[q] & \text{if } a \neq b \text{ or } d = 0 \\ \mathrm{B}_d(1, m_+)[q] & \text{if } a = b > 0 \text{ and } d > 0 \\ \mathrm{D}_n[q] & \text{if } a = b = 0 \end{cases}$$

and

$$\Omega^{\mathfrak{s}}_1 \simeq \begin{cases} C_2 & \text{if } a > 0,\ b > 0 \text{ and } a \neq b \text{ or } d > 0 \\ C_2 \times C_2 & \text{if } a = b,\ d = 0,\ \text{and } n \in 2\mathbb{Z} \\ C_4 & \text{if } a = b,\ d = 0,\ \text{and } n \in 2\mathbb{Z}+1 \\ 1 & \text{otherwise.} \end{cases}$$

As before we denote by $\mathcal{H}^{d,a,b}$ the type $\mathfrak{C}^{\mathrm{IV}}_{\mathrm{class}}$- object $\mathcal{H}^{d,a,b} \simeq \mathrm{C}_d(m_-, m_+)[q]$ which is covered by $\mathcal{H}^{\epsilon,\tilde{\mathfrak{s}},e}$. For $m_- = m_+ = 0$ we also introduce $\tilde{\mathcal{H}}^{n,0,0} \simeq \tilde{\mathrm{D}}_n[q]$. Then we have

$$\begin{cases} \mathcal{H}^{d,a,b} = \mathcal{H}^{\epsilon,\tilde{\mathfrak{s}},e} & \text{if } a \neq b \text{ or } d = 0 \\ \mathcal{H}^{d,a,b} \subset \mathcal{H}^{\epsilon,\tilde{\mathfrak{s}},e} & \text{has index two if } a = b > 0 \text{ and } d > 0 \\ \mathcal{H}^{n,0,0} \supset \tilde{\mathcal{H}}^{n,0,0} \subset \mathcal{H}^{\epsilon,\tilde{\mathfrak{s}},e} & \text{if } a = b = 0 \text{ (both inclusions have index two).} \end{cases}$$

We have, by definition of our normalizations, and using Proposition 2.5 and [9, Section 13.7],

$$\tau^{\epsilon,\mathfrak{s},e}(1)_{\mathbb{Q}} = \tau^{d,a,b}(1)_{\mathbb{Q}} = \begin{cases} 2^{-m_+} & \text{if } a = b \text{ and } d = 0,\ \text{ or if } a = b = 0 \\ 2^{1-m_+} & \text{otherwise} \end{cases}$$

where as before, $\tau^{d,a,b}$ denotes the trace of the type $\mathrm{C}_d(m_-,m_+)[q]$-algebra (an object of $\mathfrak{C}^{\mathrm{IV}}_{\mathrm{class}}$) which is spectrally covered by $\mathcal{H}^{\epsilon,\tilde{\mathfrak{s}},e}$.

The irreducible discrete series representations of $\mathcal{H}^{d,a,b}$ with central character corresponding to (u_-,u_+) are parameterized by pairs of Slooten symbols (σ_-,σ_+), denoted by $\pi_{(u_-,u_+),(\sigma_-,\sigma_+)}$. By the above, combined with Theorem 4.12 we see that

$$\mathrm{fdeg}_{\mathbb{Q}}(\pi_{(u_-,u_+),(\sigma_-,\sigma_+)})=\begin{cases}2^{-\#(u_-\cup u_+)} & \text{if } a=b \text{ and } d=0, \text{ or if } a=b=0\\ 2^{1-\#(u_-\cup u_+)} & \text{otherwise.}\end{cases}$$

As in the previous case $G=\mathrm{PCSp}_{2n}$, Proposition 4.8, (42) and (43) imply that Theorem 4.11 is true in this case iff (here λ denotes a discrete unramified Langlands parameter for G which gives rise to the pair (u_-,u_+) as in Proposition 4.8):

$$\mathrm{fdeg}_{\mathbb{Q}}(\pi^G_{\lambda,(\sigma_-,\sigma_+)})=\begin{cases}2^{-\#(u_-\cup u_+)} & \text{if } u_-=0\\ 2^{1-\#(u_-\cup u_+)} & \text{if } u_-\neq 0.\end{cases}\tag{47}$$

In the case $d=0$ we have $\mathrm{fdeg}_{\mathbb{Q}}(\pi^G_{\lambda,(\sigma_-,\sigma_+)})=\mathrm{fdeg}_{\mathbb{Q}}(\pi_{(u_-,u_+),(\sigma_-,\sigma_+)})$, and since $a=b$ is equivalent to $u_-=0$ in this case, we are done if $d=0$. Similarly, if $a\neq b$ (hence $u_-\neq 0$) there is no branching, and we are done. So from now on, we may and will assume $d>0$ and $a=b$. The case $a=b>0$ is completely analogous to what we did in the case $G=\mathrm{PCSp}_{2n}$. This leaves the case $a=b=0$. We combine results of [58, Appendix], [18, Lemma 6.10] and [56, Section 8] to derive the branching behavior of the discrete series. If $u_-=0$, then there exist two distinct discrete series representations $\pi^G_{\lambda_+,(0,\sigma_+)}$ and $\pi^G_{\lambda_-,(0,\sigma_+)}$ of $\mathcal{H}^{\epsilon,\tilde{\mathfrak{s}},e}=\mathcal{H}^{IM}(G^\epsilon)$ whose central characters are distinct (but lie in the same $X^*_{\mathrm{un}}(G^F)$-orbit), and which restrict to the same irreducible discrete series representation $\tilde{\pi}^G_{\lambda,(0,\sigma_+)}$ of $\tilde{\mathcal{H}}^{n,0,0}$. On the other hand, there also exist two irreducible discrete series characters $\pi_{(0,u_+),(0,\sigma_+)}$ and $\pi_{(0,u_+),(0,\sigma'_+)}$ of $\mathcal{H}^{n,0,0}$ which both restrict to $\tilde{\pi}^G_{\lambda,(0,\sigma_+)}$. It follows easily that $\mathrm{fdeg}_{\mathbb{Q}}(\pi^G_{\lambda_\pm,(0,\sigma_+)})=\mathrm{fdeg}_{\mathbb{Q}}(\pi_{(0,u_+),(0,\sigma_+)})=2^{-\#(u_+)}$ as desired.

If $u_-\neq 0$, and λ is an unramified discrete Langlands parameter for G corresponding to (u_-,u_+), then $\pi^G_{\lambda,(\sigma_-,\sigma_+)}$ restricts to a direct sum $\tilde{\pi}^G_{\lambda,(\sigma_-,\sigma_+,+1)}\oplus\tilde{\pi}^G_{\lambda,(\sigma_-,\sigma_+,-1)}$ of irreducible discrete series representations of $\tilde{\mathcal{H}}^{n,0,0}$. Indeed, by [58, A.13] the restriction is either irreducible or a direct sum of two irreducibles, which are moreover themselves discrete series by [18, Lemma 6.3]. Moreover it follows from [58, A.13] that if there exists a $\pi^G_{\lambda,(\sigma_-,\sigma_+)}$ with $\sigma_-\neq 0$ and $\sigma_+\neq 0$ which restricts to an irreducible in this way, then the number of irreducible discrete series representations of $\tilde{\mathcal{H}}^{n,0,0}$ with u_- and u_+ not equal to 0 is strictly less than twice the number of irreducible discrete series of the kind described above of $\mathcal{H}^{IM}(G^\epsilon)$. But this contradicts the classification of the discrete series as in [56, Section 8] (this counting argument is similar to the proof of [18, Lemma 6.10]). There are four irreducible discrete series characters $\pi_{(u_-,u_+),(\sigma_-,\sigma_+)}$, $\pi_{(u_-,u_+),(\sigma'_-,\sigma_+)}$, $\pi_{(u_-,u_+),(\sigma_-,\sigma'_+)}$ and $\pi_{(u_-,u_+),(\sigma'_-,\sigma'_+)}$ of $\mathcal{H}^{n,0,0}$, and it is easy to see that all of these restrict to irreducible discrete series characters of $\tilde{\mathcal{H}}^{n,0,0}$: Two of them will restrict to $\tilde{\pi}^G_{\lambda,(\sigma_-,\sigma_+,+1)}$, and the other two will restrict to $\tilde{\pi}^G_{\lambda,(\sigma_-,\sigma_+,-1)}$. Altogether it follows that $\mathrm{fdeg}_{\mathbb{Q}}(\pi^G_{\lambda_\pm,(\sigma_-,\sigma_+)})=$

$2\mathrm{fdeg}_{\mathbb{Q}}(\pi_{(u_-,u_+),(\sigma_-,\sigma_+)}) = 2^{1-\#(u_-\cup u_+)}$ in these cases, as desired. Using Remark 4.6 again, we see that the total number of this kind of unipotent discrete series representations equals $2^{l_-+l_+-3}$ if $l_- \neq 0$, and 2^{l_+-2} otherwise. This should correspond to the subset of the Lusztig packet associated to λ which is parameterized by the set $\mathrm{Irr}_\epsilon(A_\lambda)$ of irreducible characters of A_λ on which ${}^LZ = \Omega^*$ acts trivially. Indeed, this is half the number of one-dimensional irreducibles of A_λ.

Next, let us take the inner form G^u with $u = \eta$. The analysis is exactly the same as for $u = \epsilon$, except that now a and b are both odd. We again obtain $2^{l_-+l_+-3}$ (if $l_- \neq 0$) or 2^{l_+-2} (otherwise) unipotent discrete series representations in the Lusztig packet for λ, this times the ones parameterized by the set of irreducible characters $\mathrm{Irr}_\eta(A_\lambda)$ of A_λ on which ${}^LZ = \Omega^*$ acts as a multiple of η. The collection $\mathrm{Irr}_\epsilon(A_\lambda) \cup \mathrm{Irr}_\eta(A_\lambda)$ coincides with the collection of $2^{l_-+l_+-2}$ (if $l_- \neq 0$) (or 2^{l_+-1} if $l_- = 0$) one-dimensional irreducible representations of A_λ.

Finally consider the inner forms with $u = \rho$ or $u = \rho\eta$. These two inner forms are equivalent as rational forms, via the outer automorphism corresponding to θ, hence it suffices to consider the case $u = \rho$ only. This time the cuspidal unipotent parahoric subgroups $\mathbb{P}^\rho_{s,t}$ are given by ρ-invariant subdiagrams of type

$$\mathbb{P}^\rho_{s,t} \simeq \begin{cases} \mathrm{D}_{s^2} \cup \mathrm{D}_{s^2} \cup {}^2\mathrm{A}_{\frac{1}{2}(t^2+t)-1} & \text{if } n \text{ even} \\ {}^2\mathrm{D}_{s^2} \cup {}^2\mathrm{D}_{s^2} \cup {}^2\mathrm{A}_{\frac{1}{2}(t^2+t)-1} & \text{if } n \text{ odd} \end{cases}$$

such that $d+1 := \frac{1}{2}(n - 2s^2 - \frac{1}{2}(t^2+t) + 2) \in \mathbb{Z}_{>0}$. This corresponds to a type $\mathfrak{s} := \mathfrak{s}^\rho_{d,s,t}$ for G^ρ which is completely determined by a pair of nonnegative integers (s,t) satisfying the above inequality, and the congruences: $s \equiv n (\mathrm{mod}\ 2)$, $t \equiv 0, 3 (\mathrm{mod}\ 4)$ (if n even), and $t \equiv 1, 2 (\mathrm{mod}\ 4)$ (if n odd). The corresponding affine Hecke algebra $\mathcal{H}^{\rho,\tilde{\mathfrak{s}},e}$ is of always type $C_d(m_-,m_+)[q^2]$, with $m_+ = \frac{1}{4}(1+2t+4s)$ and $m_- = \frac{1}{4}|1+2t-4s|$. We have

$$\Omega_1^{\mathfrak{s}} = \begin{cases} \Omega & \text{if } s > 0 \text{ or } d = 0 \\ \langle\rho\rangle \simeq C_2 & \text{if } s = 0 \text{ and } d > 0. \end{cases} \tag{48}$$

Using 2.5 and [9, Section 13.7], the rational factor $\tau^{\rho,\mathfrak{s},e}(1)_{\mathbb{Q}}$ of $\tau^{\rho,\mathfrak{s},e}(1)$ equals

$$\tau^{\rho,\mathfrak{s},e}(1)_{\mathbb{Q}} = 2^{-s-1} = \begin{cases} 2^{-\frac{1}{2}(m_++m_-+2)} & \text{if } \epsilon_- \neq \epsilon_+ \\ 2^{-\frac{1}{2}(m_+-m_-+2)} & \text{if } \epsilon_- = \epsilon_+. \end{cases}$$

Using Theorem 4.14, we obtain discrete series representations $\pi^\alpha_{(\pi_-,\pi_+),extra}$ with rational parts of formal degrees equals $\mathrm{fdeg}_{\mathbb{Q}}(\pi^\alpha_{(\pi_-,\pi_+),extra}) = 2^{\frac{1}{2}(l_-+l_+-2)-\#(u_-\cup u_+)}$ (in all cases), where α denotes an irreducible character of $\Omega_1^{\mathfrak{s}}$. In view of Proposition 4.8 and (47), this is indeed the rational factor of the formal degree of the two elements of the Lusztig packet attached to the Langlands parameter λ on which ${}^LZ \subset A_\lambda$ acts by ρ times the identity, as predicted by (43). As to the numerology of counting the number of such irreducible representations in a Lusztig packet attached to a unipotent discrete Langlands parameter λ for G: Let us write (u_-, u_+) for the (ordered) pair of unipotent partitions attached to λ (these are partitions with odd, distinct parts such that $|u_-| + |u_+| = 2n$). If $u_- \neq 0$ and $u_- \neq u_+$, then we have two such packets (for the pairs (u_-, u_+) and (u_+, u_-) which contain a discrete series representation with the same q-rational factor. According to Proposition 4.8 both these packets contain 2 irreducibles on which ${}^L Z$ acts as a multiple of ρ (and also two where ${}^L Z$ acts as multiple of $\rho\eta$) (together these are the four irreducibles in each of these packets which are not one-dimensional). This matches the "Hecke side", since we have (by (48)) that $\mathcal{H}^{\rho,\mathfrak{s}}$ is either a direct sum of four copies of $\mathcal{H}^{\rho,\tilde{\mathfrak{s}},e}$, each contributing one irreducible discrete series with the desired q-rational factor in the formal degree (if $s \neq 0$, or equivalently $m_- \neq m_+$) or of two such copies (if $s = 0$, or equivalently $m_- = m_+$). But in the latter case, each of these copies of $\mathcal{H}^{\rho,\tilde{\mathfrak{s}},e}$ contributes two such irreducible discrete series (whose central characters are mapped by the STM to (u_-, u_+) and (u_+, u_-) respectively). If $u_- = u_+$, then necessarily $m_- = m_+$, and the two copies of $\mathcal{H}^{\rho,\tilde{\mathfrak{s}},e}$ contribute each one discrete series to the packet associated to λ, corresponding to the two irreducibles of A_λ on which ${}^L Z$ acts as ρ. Finally we have the case $u_- = 0$. In this case there are four distinct discrete Langlands parameters $\lambda_1 = \lambda,\ \lambda_2,\ \lambda_3,\ \lambda_4$ which share the same q-rational factor in the formal degree, and each of the four corresponding Lusztig packets should have one member associated to the single irreducible of A_{λ_i} on which ${}^L Z$ acts by ρ (according to Proposition 4.8). Hence in all cases the Hecke algebra side and the L-packet side indeed match. This finishes the case $G = \mathrm{P}(\mathrm{CO}^0_{2n})$.

The last case to consider is the non-split quasisplit orthogonal group $\mathrm{P}((\mathrm{CO}^*)^0_{2n+2})$. Now we have $u \in \Omega/(1-\theta)\Omega = \Omega/\langle\eta\rangle \simeq \langle\overline{\rho}\rangle \simeq C_2$. We have $\mathcal{H}^{IM}(G) = \mathrm{C}_n(1,1)[q]$. The conjugacy classes of parahoric subgroups $\mathbb{P}^{d,a,b}$ which support a (unique) cuspidal unipotent representation are parametrized by ordered pairs (a, b) with $a, b \in \mathbb{Z}_{\geq 0}$, with a even and b odd, and such that $d = n + 1 - a^2 - b^2 \geq 0$. The parahoric $\mathbb{P}^{d,a,b}$ is of type $\mathrm{D}_{a^2} \cup {}^2\mathrm{D}_{b^2}$. The corresponding cuspidal unipotent type is denoted by $\mathfrak{s} = \mathfrak{s}^{d,a,b}$. We have $\mathcal{H}^{\epsilon,\tilde{\mathfrak{s}},e} = \mathrm{C}_d(m_-, m_+)[q]$, with $m_+ = a + b$ and $m_- = |a - b|$. Furthermore, $\Omega_1^{\epsilon,\mathfrak{s},\theta} = C_2$ (if $a > 0$ or $d = 0$) or $= 1$ (if $a = 0$ and $d > 0$), implying that $\tau^{\epsilon,\mathfrak{s},e}(1)_{\mathbb{Q}} = 2^{1-m_+}$ (in all cases).

Let λ be a discrete unramified Langlands parameter for G. According to [61], in the notation of (14), we have $C_{G^\vee}(\lambda(\mathrm{Frob} \times \mathrm{id}))$ is the connected cover in Spin_{2n+2} of $\mathrm{SO}_{2n_-+1} \times \mathrm{SO}_{2n_++1}$ (with $n_- + n_+ = n$), and the $G^\vee$-orbits of such λ correspond bijectively to ordered pairs (u_-, u_+) where $u_\pm$ is a distinguished unipotent class in $\mathrm{SO}_{2n_\pm+1}$. Note that this means that $u_\pm \vdash 2n_\pm + 1$ has odd, distinct parts.

Let (σ_-, σ_+) be a Slooten symbol for the parameters (m_-, m_+) corresponding to the pair (λ_-, λ_+), and let $\pi_{(u_-,u_+),(\sigma_-,\sigma_+)}$ be the correspond discrete series representation of $\mathcal{H}^{\epsilon,\tilde{\mathfrak{s}},e}$. Then, Theorem 4.12 implies that $\mathrm{fdeg}_{\mathbb{Q}}(\pi_{(u_-,u_+),(\sigma_-,\sigma_+)}) = 2^{1-\#(u_-\cup u_+)}$. It easily follows that this agrees with (43). The number of such irreducible discrete series equals $2^{l_-+l_+-2}$, as expected by Proposition 4.8.

Let us now consider $u = \overline{\rho}$. Now the cuspidal unipotent parahoric subgroups $\mathbb{P}^{\overline{\rho}}_{s,t}$ are given by $\overline{\rho}$-invariant subdiagrams of type

$$\mathbb{P}^{\rho}_{s,t} \simeq \begin{cases} \mathrm{D}_{s^2} \cup \mathrm{D}_{s^2} \cup {}^2\mathrm{A}_{\frac{1}{2}(t^2+t)-1} & \text{if } n \text{ even} \\ {}^2\mathrm{D}_{s^2} \cup {}^2\mathrm{D}_{s^2} \cup {}^2\mathrm{A}_{\frac{1}{2}(t^2+t)-1} & \text{if } n \text{ odd} \end{cases}$$

such that $d + 1 := \frac{1}{2}(n - 2s^2 - \frac{1}{2}(t^2 + t) + 3) \in \mathbb{Z}_{>0}$. This corresponds to a type $\mathfrak{s} := \mathfrak{s}^{\rho}_{d,s,t}$ for G^{ρ} which is completely determined by a pair of nonnegative integers (s, t) satisfying the above inequality, and the congruences: $s \equiv n(\mathrm{mod}\ 2)$, $t \equiv 1, 2(\mathrm{mod}\ 4)$ (if n even), and $t \equiv 0, 3(\mathrm{mod}\ 4)$ (if n odd). The corresponding affine Hecke algebra $\mathcal{H}^{\rho,\tilde{\mathfrak{s}},e}$ is of always type $C_d(m_-, m_+)[q^2]$, with $m_+ = \frac{1}{4}(1 + 2t + 4s)$ and $m_- = \frac{1}{4}|1 + 2t - 4s|$. We have

$$\Omega^{\mathfrak{s},\theta}_1 = \begin{cases} \langle\eta\rangle \simeq C_2 & \text{if } s > 0 \text{ or } d = 0 \\ 1 & \text{if } s = 0 \text{ and } d > 0 \end{cases}$$

and we get

$$\tau^{\rho,\mathfrak{s},e}(1)_{\mathbb{Q}} = 2^{-s} = \begin{cases} 2^{-\frac{1}{2}(m_+ + m_-)} & \text{if } \epsilon_- \neq \epsilon_+ \\ 2^{-\frac{1}{2}(m_+ - m_-)} & \text{if } \epsilon_- = \epsilon_+. \end{cases}$$

Hence, using Theorem 4.14, the extra special STM $\mathcal{H}^{\rho,\tilde{\mathfrak{s}},e} \rightsquigarrow \mathcal{H}^{IM}(G)$ yields one additional discrete series representation $\pi_{(\pi_-,\pi_+),extra}$ added to the Lusztig packet associated to λ, whose formal degree satisfies $\mathrm{fdeg}_{\mathbb{Q}}(\pi_{(\pi_-,\pi_+),extra}) = 2^{\frac{1}{2}(l_- + l_+) - \#(u_- \cup u_+)}$, as desired in view of Proposition 4.8. □

4.6.2 Unipotent representations for inner forms of SO_{2n+1} In these cases, a unipotent affine Hecke algebra is always spectrally isomorphic to a direct sum of finitely many copies of objects of $\mathfrak{C}^{\mathrm{II}}_{\mathrm{class}}$. The treatment of these cases is analogous to the symplectic and even orthogonal cases discussed in the previous paragraph, but in all aspects much simpler (no branching phenomena, no extraspecial STM's). We will content ourselves to give the results only.

We have $\Omega = \{\epsilon, \eta\} \simeq C_2$, and $\mathcal{H}^{IM}(G^{\epsilon})$ is of type $\mathrm{C}_n(\frac{1}{2}, \frac{1}{2})[q]$. The conjugacy classes of parahoric subgroups $\mathbb{P}^{d,a,b}$ of G supporting a (unique) cuspidal unipotent representation are parametrized by ordered pairs (a, b) with $a, b \in \mathbb{Z}_{\geq 0}$, with a even, and such that $d = n - a^2 - (b^2 + b) \geq 0$. The parahoric $\mathbb{P}^{d,a,b}$ is a type $\mathrm{D}_{a^2} \cup \mathrm{B}_{b^2+b}$. The corresponding cuspidal unipotent type is denoted by $\mathfrak{s} = \mathfrak{s}^{d,a,b}$. We have $\mathcal{H}^{\epsilon,\tilde{\mathfrak{s}},e} = \mathrm{C}_d(m_-, m_+)[q]$, with $m_+ = \frac{1}{2} + a + b$ and $m_- = |\frac{1}{2} - a + b|$. Furthermore, $\Omega^{\epsilon,\mathfrak{s},\theta}_1 = C_2$ (if $a > 0$ or $d = 0$) or $= 1$ (if $a = 0$ and $d > 0$), implying that $\tau^{\epsilon,\mathfrak{s},e}(1)_{\mathbb{Q}} = 2^{\frac{1}{2} - m_+}$ (in all cases). For the nontrivial inner form G^{η} of G, the formulas are the same except that now a is odd, and $\mathbb{P}^{d,a,b}$ has type ${}^2\mathrm{D}_{a^2} \cup \mathrm{B}_{b^2+b}$.

Now an orbit of discrete unipotent Langlands parameters λ for G corresponds to an ordered pair (u_-, u_+) of unipotent partitions with $u_{\pm} \vdash 2n_{\pm}$ such that $n_- + n_+ = n$, where $u_{\pm}$ consists of distinct, even parts.

The discrete series representations of $\mathcal{H}^{\epsilon,\tilde{\mathfrak{s}},e} = \mathrm{C}_d(m_-, m_+)[q]$ are parameterized by a pair of Slooten symbols (σ_-, σ_+) for such pairs (u_-, u_+), at the parameter pair (m_-, m_+). The ordered pair (σ_-, σ_+) corresponds to an ordered pair of partitions (π_-, π_+) with $\pi_\pm \vdash n_\pm$. Let us denote this discrete series representation of $\mathrm{C}_d(m_-, m_+)[q]$ by $\delta_{(\pi_-,\pi_+)}$. By Theorem 4.13 we arrive at

$$\mathrm{fdeg}_{\mathbb{Q}}(\pi_{(u_-,u_+),(\sigma_-,\sigma_+)}) = 2^{-\#(u_-\cup u_+)}. \tag{49}$$

It is easy to check that this matches (43) (see e.g. [15, Corollary 6.1.6]).

4.6.3 The example of type $^3D_4$ Let G be the group of type $^3\mathrm{D}_4$ defined over a non-archimedean local field k. The group G is quasisplit, and the dual L-group is isomorphic to $^LG := \langle\theta\rangle \ltimes G^\vee$ where $G^\vee = \mathrm{Spin}(8)$ and where θ is an outer automorphism of order 3. Hence $^LZ = 1$, and G has no nontrivial inner forms. There are two cuspidal unipotents called $^3\mathrm{D}_4[1]$ and $^3\mathrm{D}_4[-1]$ (cf. [9], section 13.7).

The image $\phi(F) = s\theta$ of the Frobenius element under a discrete unramified Langlands parameter ϕ is an isolated semisimple automorphism. Via the action of $\mathrm{Int}(G^\vee)$ it is conjugate to a semisimple element of the form θs_i with s_i a vertex of C_θ (cf. [20]). In the case at hand, we label the nodes of the twisted affine root diagram according to [20, Section 4.4], and we have to consider θs_0, θs_1 and θs_2.

We have $\mathcal{H}^{IM}(G) = \mathrm{G}_2(3,1)[q]$, normalized by $\tau(1) := [3]_q^{-1}(v-v^{-1})^{-2}$ according to (25). The W_0-orbit space of the character torus T of the root lattice X of type G_2 can be identified [3] with the space of $\mathrm{Int}(G^\vee)$-orbits of semisimple classes of ^LG of the form θg, via the map $T \ni t \to \theta t$. In this way we will identify, as usual, the space of central characters of affine Hecke algebra $\mathcal{H}^{IM}(G) = G_2(3,1)[q]$ and the space of semisimple $\mathrm{Int}(G^\vee)$-orbits of this form of ^LG. The Hecke algebra $\mathcal{H}^{IM}(G)$ has two orbits of real residual points $W_0 r_{0,reg}$ and $W_0 r_{0,\mathrm{sub}}$, and two nonreal ones $W_0 r_1$ and $W_0 r_2$ (using the same numbering of the nodes of the diagram as before). At each residual point of $G_2(m_l, m_s)[q]$ at the parameter value $(m_l, m_s) = (3, 1)$, the number of irreducible discrete series characters supported at this point is equal to the number of generic residual points which specialize at $(3, 1)$ to the given residual point. This number is always 1, except for $W_0 r_{0,\mathrm{sub}}$, where it is equal to two [56].

We can and will baptise these orbits of generic residual points $W_0 r$, using Kazhdan–Lusztig parameters for the discrete series of $G_2(1,1)[q]$, by an irreducible representation of A_λ, where λ is the Langlands parameter of the split group of type G_2. The subregular unipotent orbit of G_2 gives rise to a unipotent discrete Langlands parameter $\lambda = \lambda_{\mathrm{sub}}$ of $^3\mathrm{D}_4$ with $A_\lambda = S_3$. Its "weighted Dynkin diagram" is $r_{0,\mathrm{sub}}$. The two orbits of generic residual points of the generic Hecke algebra of type G_2 which are confluent at $(1, 1)$ are also confluent at $(3, 1)$. By the above, we call these two orbits of generic residual points $W_0 r_{\mathrm{sub,triv}}$ and $W_0 r_{\mathrm{sub},\sigma}$, where σ is the two dimensional irreducible character of S_3. The orbit of generic points $W_0 r_{\mathrm{sub,triv}}$ represents a generic discrete series character of degree 3, which has generic formal degree with rational constant factor $\frac{1}{2}$. The other orbit of generic residual points $W_0 r_{\mathrm{sub},\sigma}$ has degree 1, and generic rational constant 1.

At the confluence of these two generic residual points at parameter $(1, 1)$, we get in the limit an additional constant factor $\frac{3}{1}$ for $W_0 r_{\mathrm{sub,triv}}$ leading to the well known equal parameter case of Theorem 4.11 at the subregular unipotent orbit for split G_2 (cf. [59]). At the confluence point for the parameters (3, 1) the rational constants do not change, however. Thus together with the cuspidal character ${}^3D_4[1]$ we get a packet Π_λ for $\lambda = \lambda_{\text{sub}}$ consisting of three representations, naturally parameterized by the characters of $A_\lambda = S_3$, whose formal degrees have rational constant $\frac{1}{2}$ (for the cuspidal ${}^3D_4[1]$ corresponding to the "missing representation" $sign$ of $A_\lambda = S_3$, and for the generic discrete series character associated to $W_0 r_{\text{sub,triv}}$ evaluated at the parameter value (3, 1)), and rational constant 1 (for the generic discrete series $W_0 r_{\text{sub},\sigma}$ evaluated at (3, 1)).

For the regular parameter of $G^\vee_{\theta s_1}$, we get two discrete series characters, namely the cuspidal one ${}^3D_4[-1]$ and the Iwahori spherical one. Both have $\frac{1}{2}$ as a rational constant factor.

Finally, at the regular parameter of $G^\vee_{\theta s_2}$, we have one Iwahori spherical discrete series representation, with rational constant 1.

These constants are clearly compatible with Theorem 4.11. Namely, consider (43). For a discrete Langlands parameters λ with $\theta s_i = \lambda(F)$ and such that $u := \lambda\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is regular within the connected reductive group $G^\vee_{\theta s_i}$, this follows because $C^F_\lambda = 1$ in such a case (this is obvious for $i = 0$ and $i = 2$ since then $A_\lambda = 1$, and for $i = 1$ we see that u is the distinguished element [5, 3] in Spin(8) by the table of [9, p. 397], whence $M^0_\lambda = 1$), and hence $A_\lambda \approx (\pi_0(M_\lambda))^F$ is isomorphic to $Z(G^\vee_{\theta s_i})/Z({}^L G) = Z(G^\vee_{\theta s_i})$ (by [59, Section 6]). This yields the result for all cases except λ_{sub}. For this case we remark that the image of the subregular unipotent of G_2 in Spin(8) is a unipotent class of Spin(8) with elementary divisors [1, 1, 3, 3]. Hence M^0_λ is a two-dimensional torus, on which F acts as a rotation of order 3. Thus C^F_λ is cyclic of order 3, and $(\pi_0(M_\lambda))^F \approx C_2$. This indeed yields the constants we just computed.

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